GEOMETRICAL CONSTRUCTIONS 1

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INTRODUCTION

Drawing instruments (necessary to this and further courses)

- loose-leaf white papers of the size A4
- 2 mechanical (propelling) pencils
  thickness/hardness: 0.35/H (or HB) and 0.5/2B
- eraser
- rulers: 1 straight and 2 set squares
- pair of compasses (with joint for pen)
- set of coloured pencils (min. 12 pcs)
- sharpener
- drawing board (size A3)

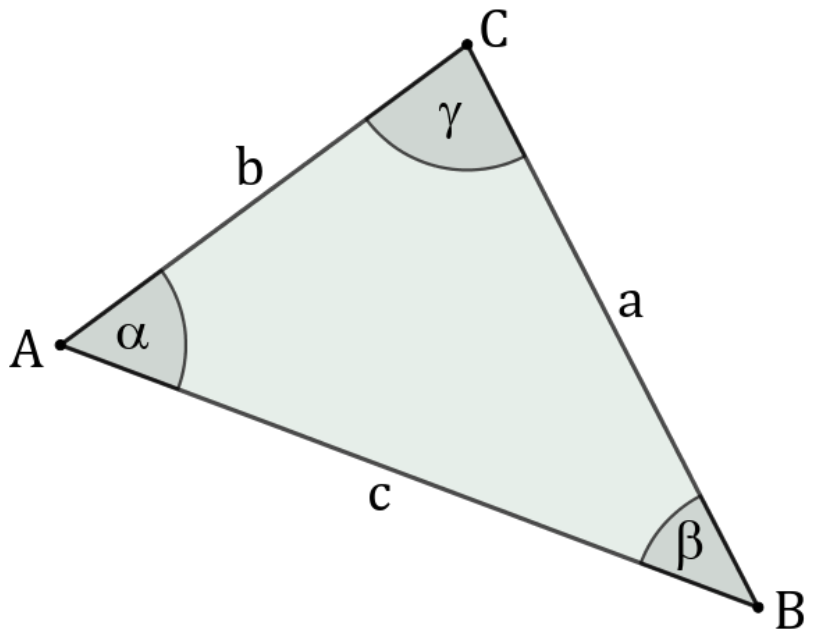
Basic constructions

- drawing parallel lines (by using set squares)
- drawing perpendicular lines (by using set squares)
- finding the midpoint of an segment
- copying an angle
- bisecting an angle

✍ lecture & practical (This sign means that you must take *your own handwritten notes and drawings* on lecture/practical.)

TRIANGLES AND THEIR PROPERTIES

A *triangle* is plane figure that has 3 vertices/vertexes and 3 sides (denoted by eg. △ABC)

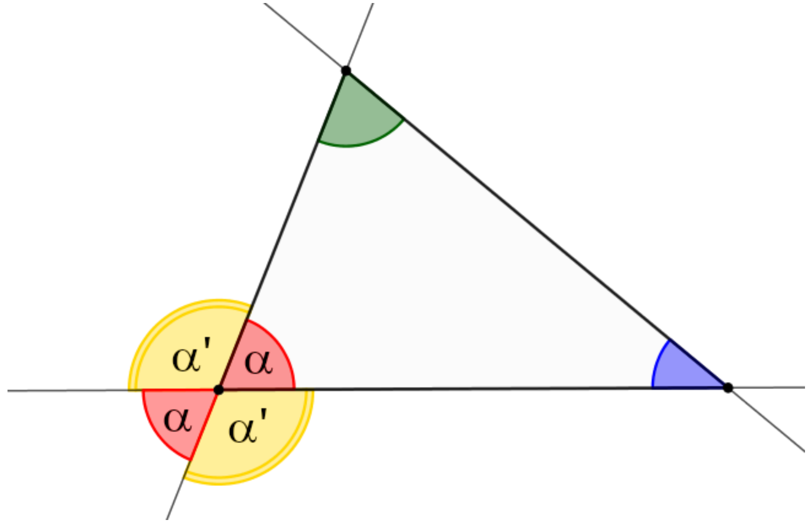


Vertices: A, B, C (capital letters)

Sides: a, b, c (small letters)

(Interior) angles: α, β, γ (small Greek letters)

Exterior angles: eg. α’



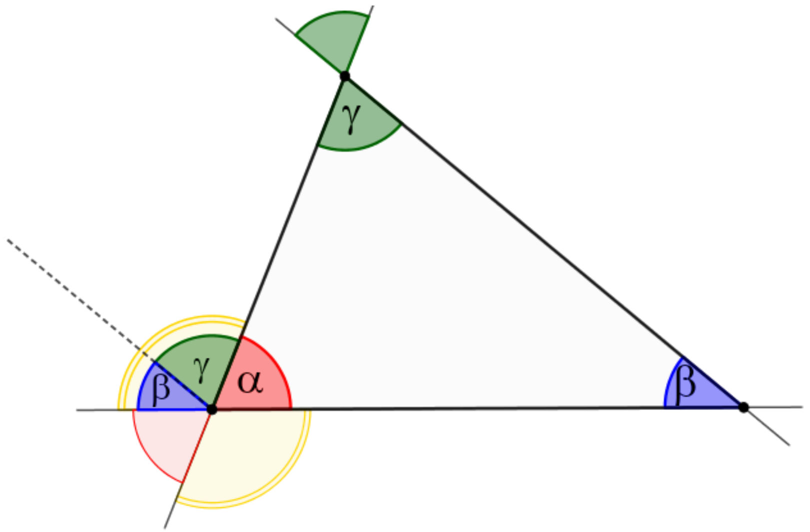
$\alpha + \alpha' = 180°$

$\beta + \beta' = 180°$

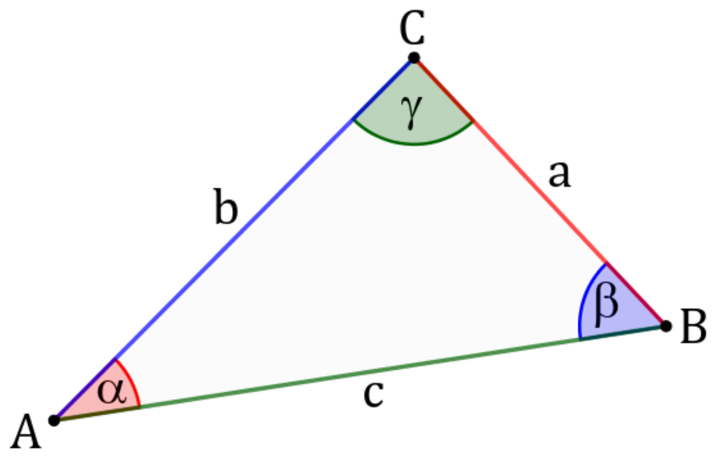
$\gamma + \gamma' = 180°$

Essential theorems about triangles

$\beta + \gamma = \alpha'$ $\alpha + \gamma = \beta'$ $\alpha + \beta = \gamma'$



The sum of the measures of the interior angles of a triangle is always 180 degrees: $\alpha + \beta + \gamma = 180°$



Triangle inequalities: $a < b + c$ $b < a + c$ $c < a + b$

$a < b < c \Leftrightarrow \alpha < \beta < \gamma$

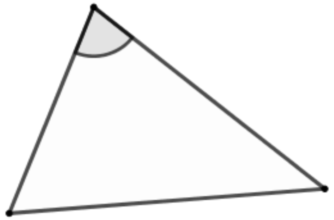
("Bigger side and bigger angle are opposite each other.")

Types of triangles

By interior angles

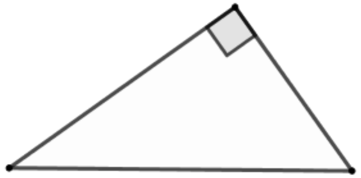
Acute triangle

The biggest interior angle is an acute angle (< 90°).



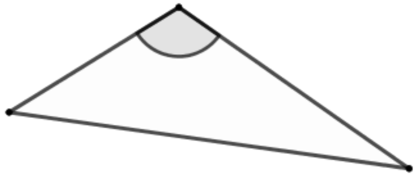
Right triangle

The biggest interior angle is a right angle (= 90°).



Obtuse triangle

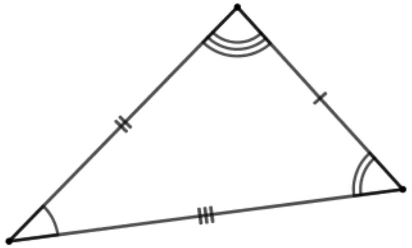
The biggest interior angle is an obtuse angle (> 90°).



By lengths of sides

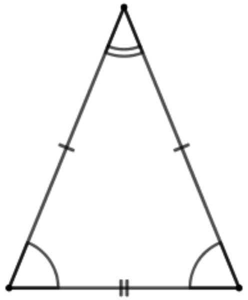
Scalene triangle

has all its sides of different lengths.



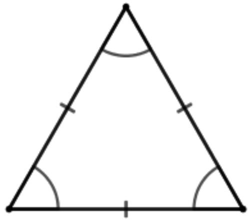
Isosceles triangle

has two sides of equal length.



Equilateral triangle

has all sides the same length.



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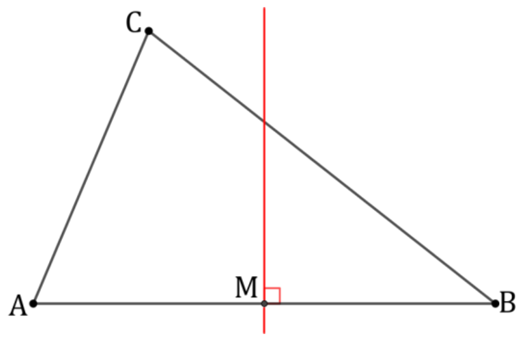
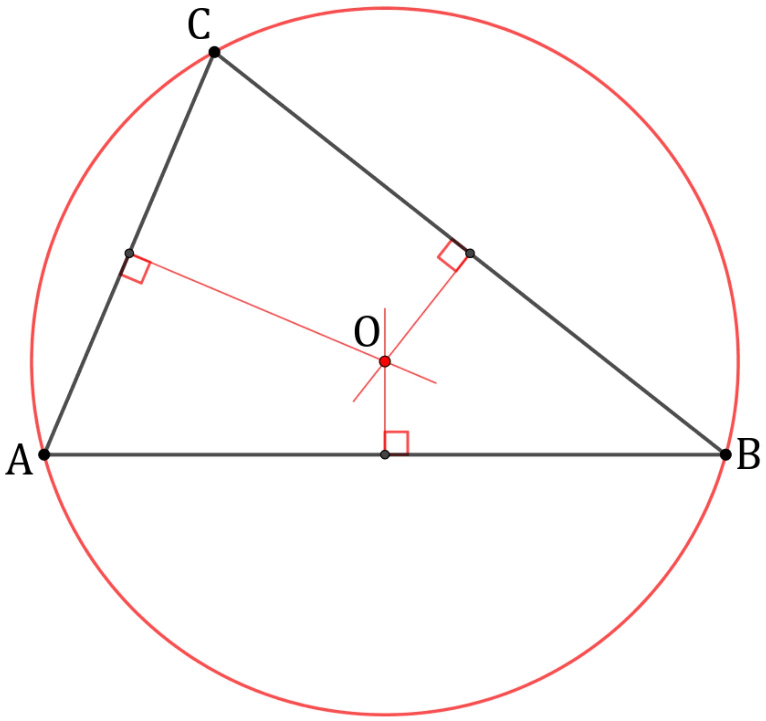
BASIC THEOREMS ABOUT TRIANGLES – 1

Perpendicular bisector
is perpendicular to a segment at its midpoint.



Circumscribed circle or circumcircle
is a circle which passes through all the vertices of the triangle.

Circumcenter
is the center of the triangle's circumcircle.

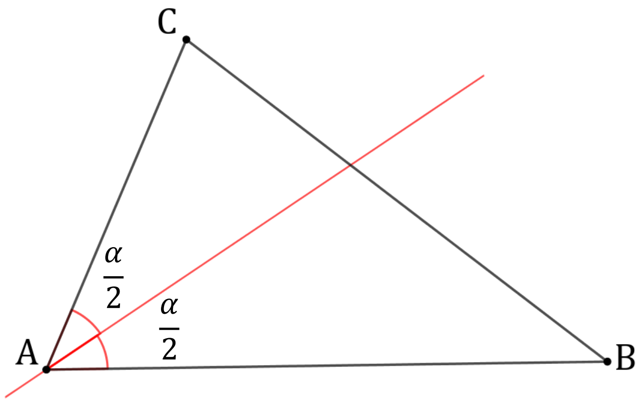


> The perpendicular bisectors of a triangle meet at the circumcenter of this triangle.

✍ lecture & practical

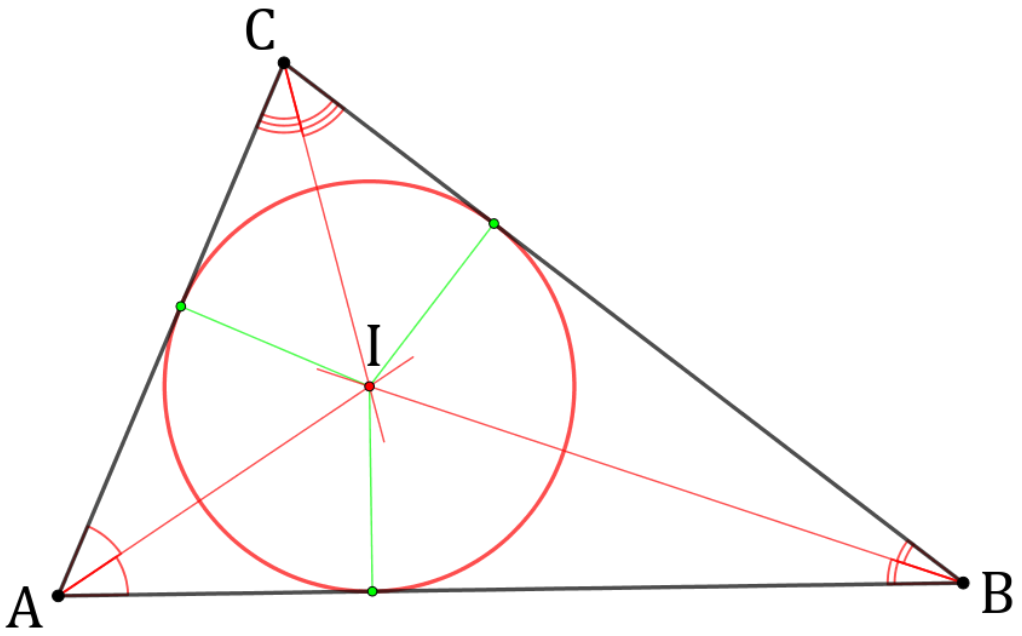
Basic theorems about triangles – 2a

(Interior) angle bisector
divides the angle into two angles with equal measures.



Inscribed circle or incircle
touches (is tangent to) the triangle’s three sides.

Incenter
is the center of the triangle’s incircle.



The interior angle bisectors of a triangle meet at the incenter of this triangle.

✍ lecture & practical

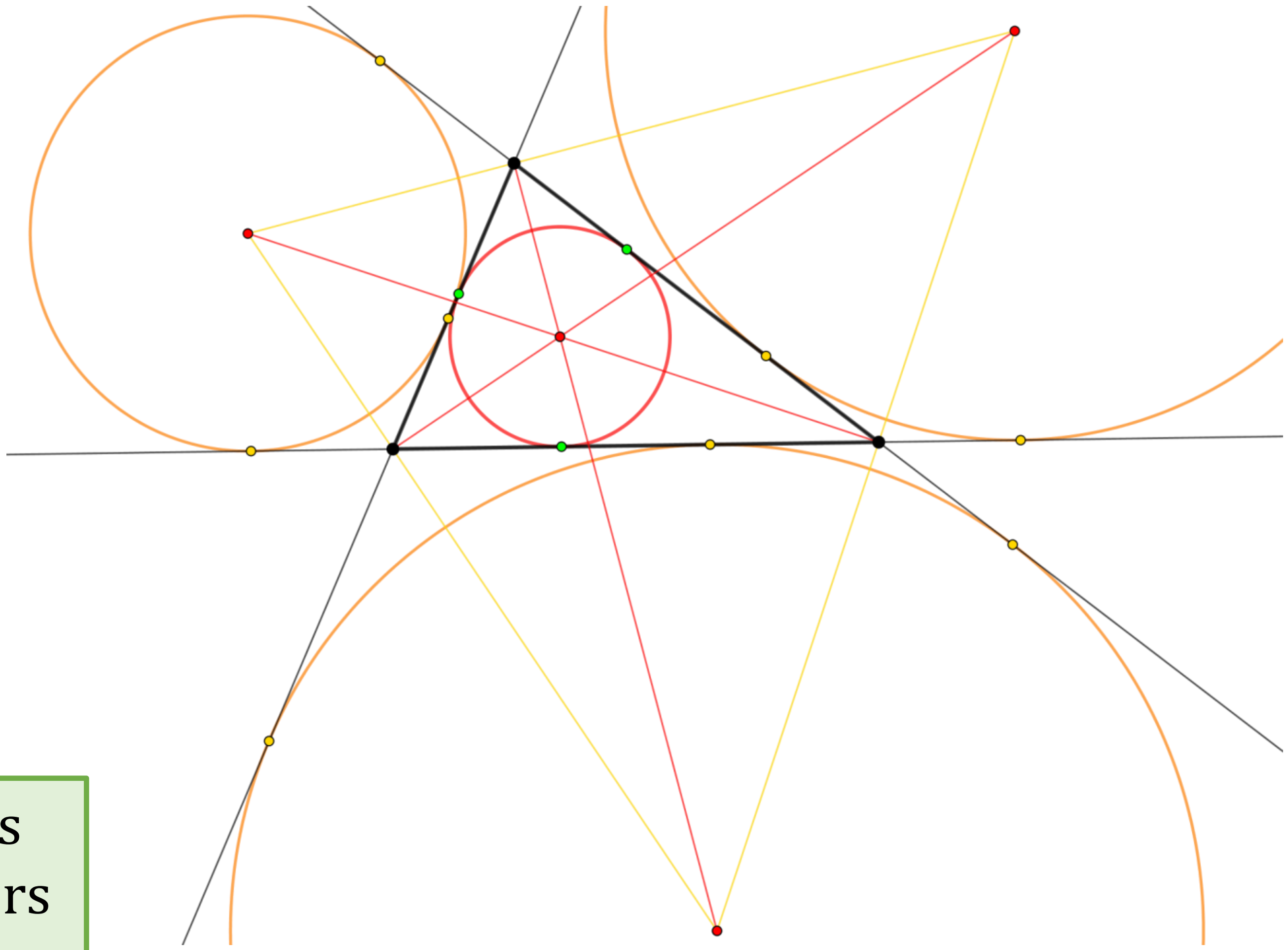
Basic theorems about triangles – 2b

Remark

Exterior angle bisector ⟶

Exscribed circle or excircle
touches the triangle’s three sides
as straight lines.
(Number of excircles: 3)

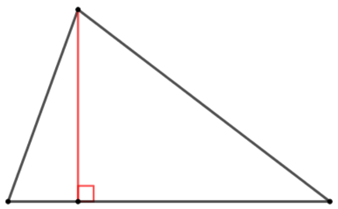
Excenter
is the center of the triangle’s excircle.

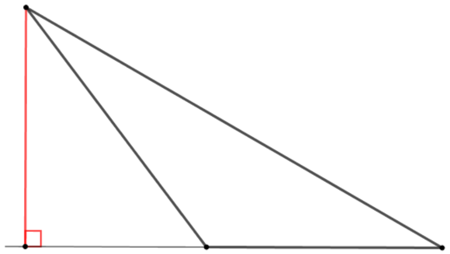


Two exterior and one interior angle bisectors of a triangle meet at one of the three excenters of the triangle.

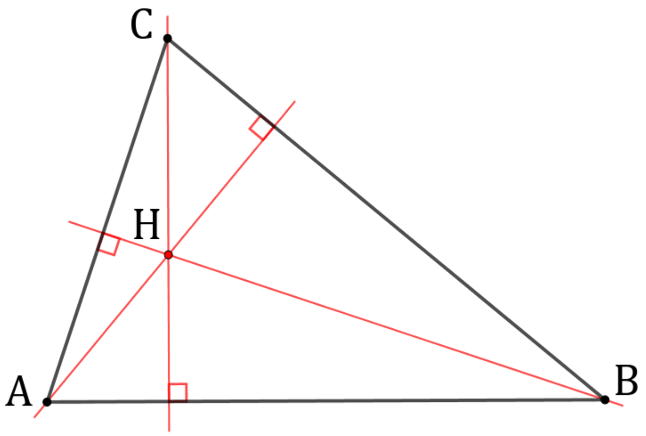
Basic theorems about triangles – 3

Altitude
is a segment (or line) through a vertex and perpendicular to the side opposite the vertex.
(Number of altitudes of a triangle: 3)





Three altitudes of a triangle intersect in a single point, called the *orthocenter* of the triangle.



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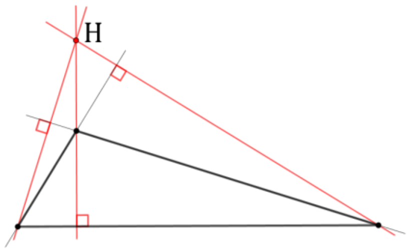
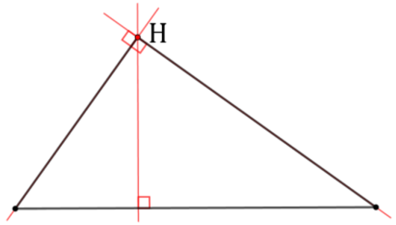
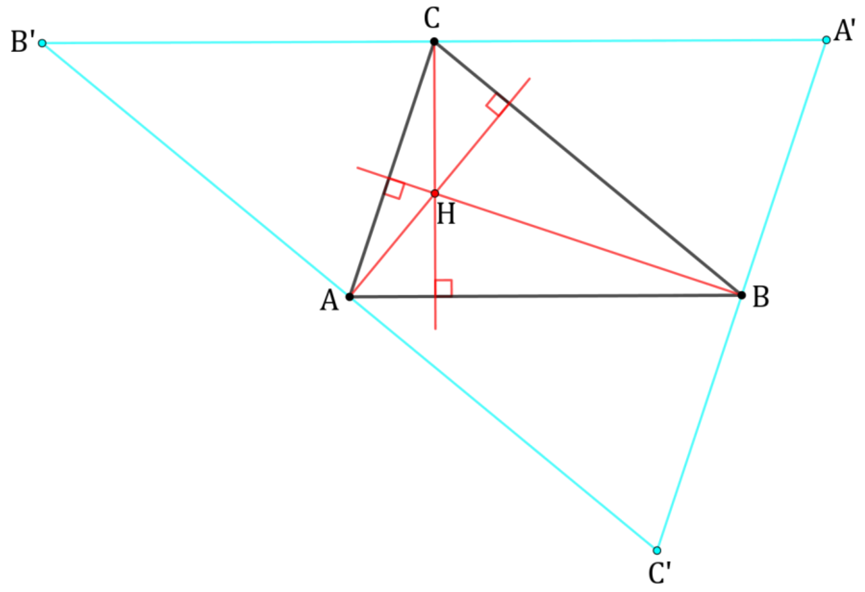
Point H lies outside △ABC if and only if △ABC is obtuse.
If △ABC is a right triangle, point H coincides with the vertex at the right angle.





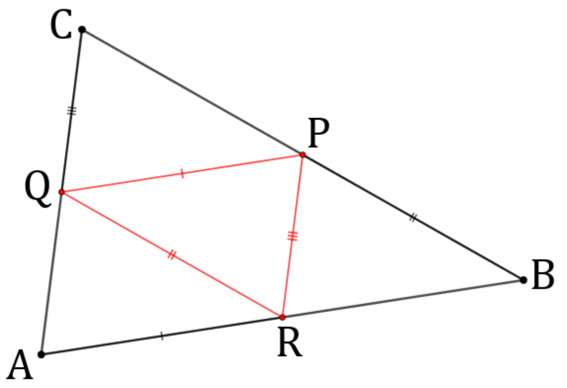
Basic theorems about triangles – 4

Sketch of the proof existing of orthocenter



Midsegment (midline)
joins the midpoints of two sides of the triangle.

$AB \parallel QP$	$BC \parallel RQ$	$AC \parallel RP$
$QP = \frac{AB}{2}$	$RQ = \frac{BC}{2}$	$RP = \frac{AC}{2}$



$\triangle PQR$ is called medial triangle.

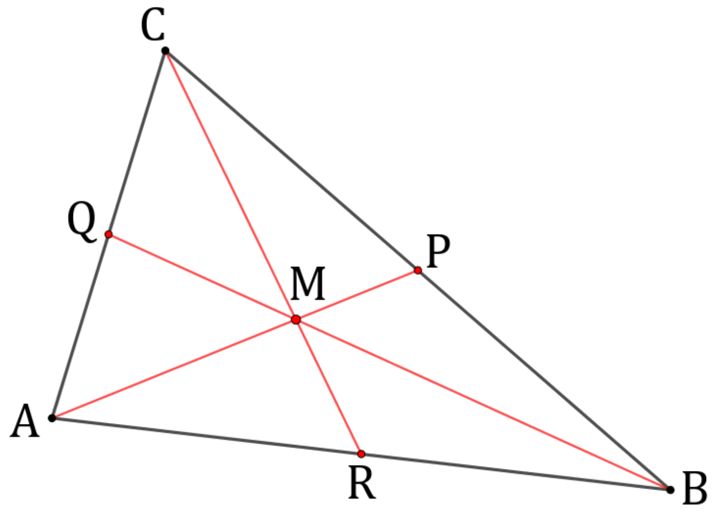
Basic theorems about triangles – 5

Median
connects a vertex with the midpoint of the opposite side.
(Number of medians of a triangle: 3)

Three medians of a triangle intersect in a single point, called the *centroid* of the triangle.

Moreover,

$$\mathrm{AM} = 2 \cdot \mathrm{MP} \qquad \mathrm{BM} = 2 \cdot \mathrm{MQ} \qquad \mathrm{CM} = 2 \cdot \mathrm{MR}$$



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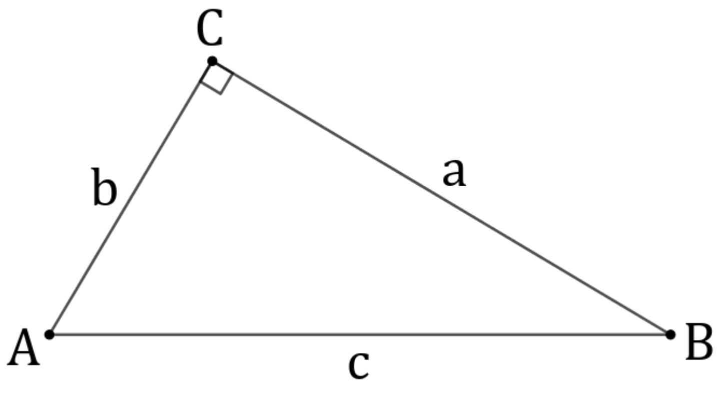
Basic theorems about triangles – 6

Let $\triangle ABC$ be a right triangle.

Legs of the triangle – a, b
Hypotenuse of the triangle – c



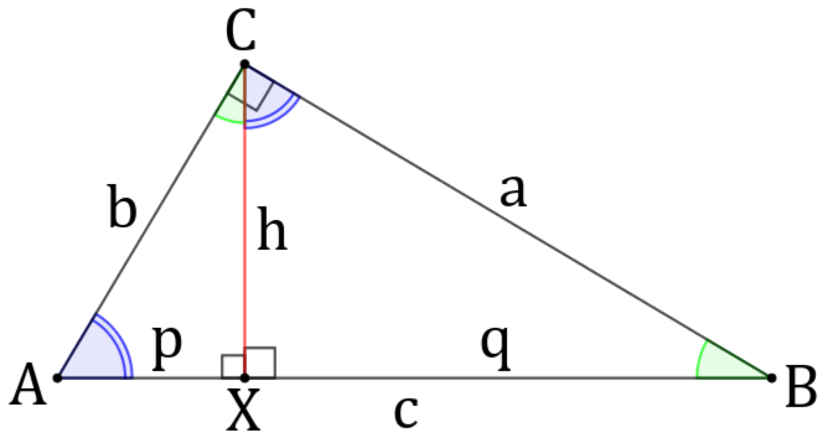
Pythagoras' theorem

$\triangle ABC$ is a right triangle $\quad \Leftrightarrow \quad a^2 + b^2 = c^2$

Let $\triangle ABC$ be a right triangle.

Then

$h^2 = p \cdot q \qquad a^2 = c \cdot q \qquad b^2 = c \cdot p$



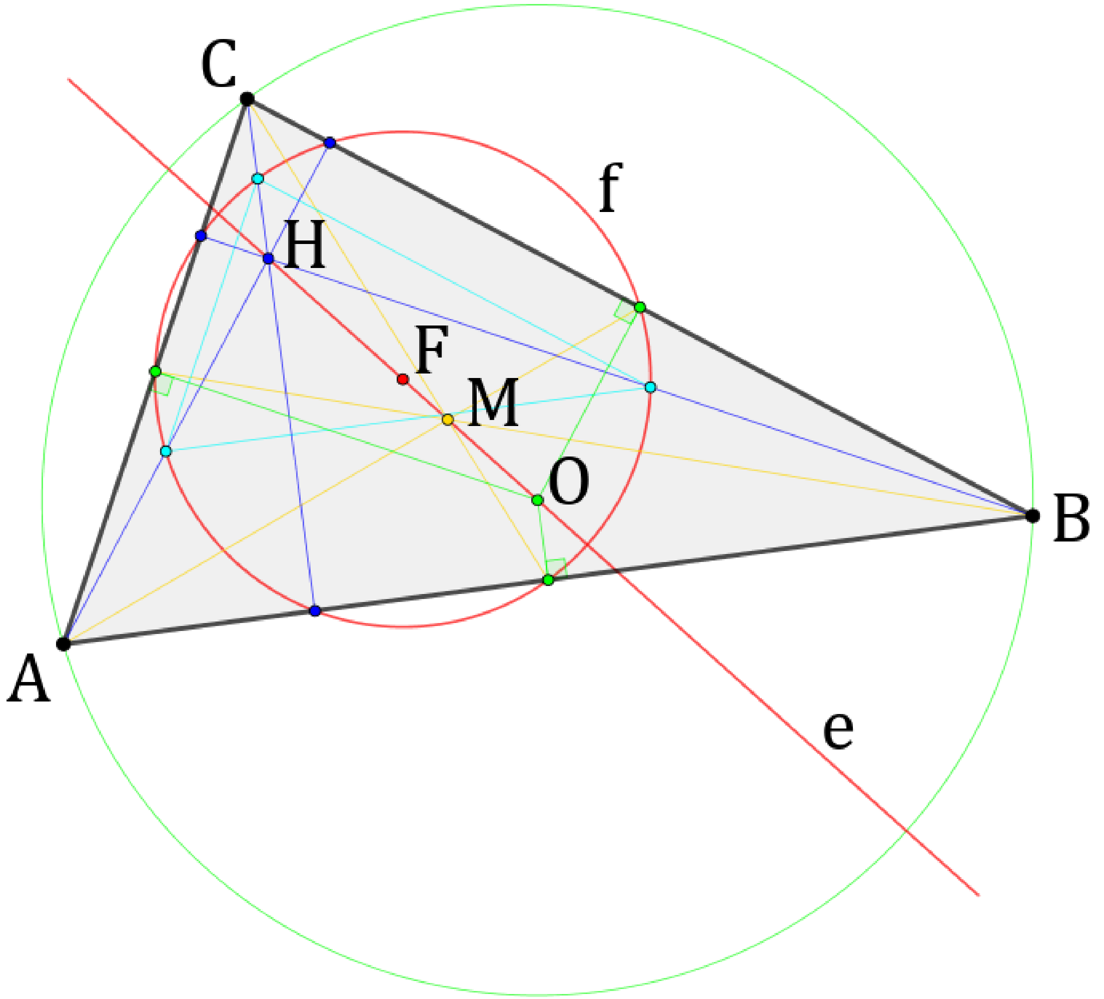
Proof: $\longrightarrow$ Similarities

Interesting theorems about triangles

H – orthocenter of △ABC
M – centroid of △ABC
O – circumcenter of △ABC

O, H and M are collinear.
This straight line is called *Euler line*.
The ratio of HM to MO is 2:1 .

The midpoints of the sides of the triangle,
the foots of the altitudes, and
the midpoints of segments AH, BH, and CH
lie on a circle called
nine-point circle (or Feuerbach's circle).
(HF = FO)



CIRCLES AND RELATED TERMS

Circle is the set of all points in a plane that are at a given distance from a given point.

O – center

r – radius

d – diameter

c – chord

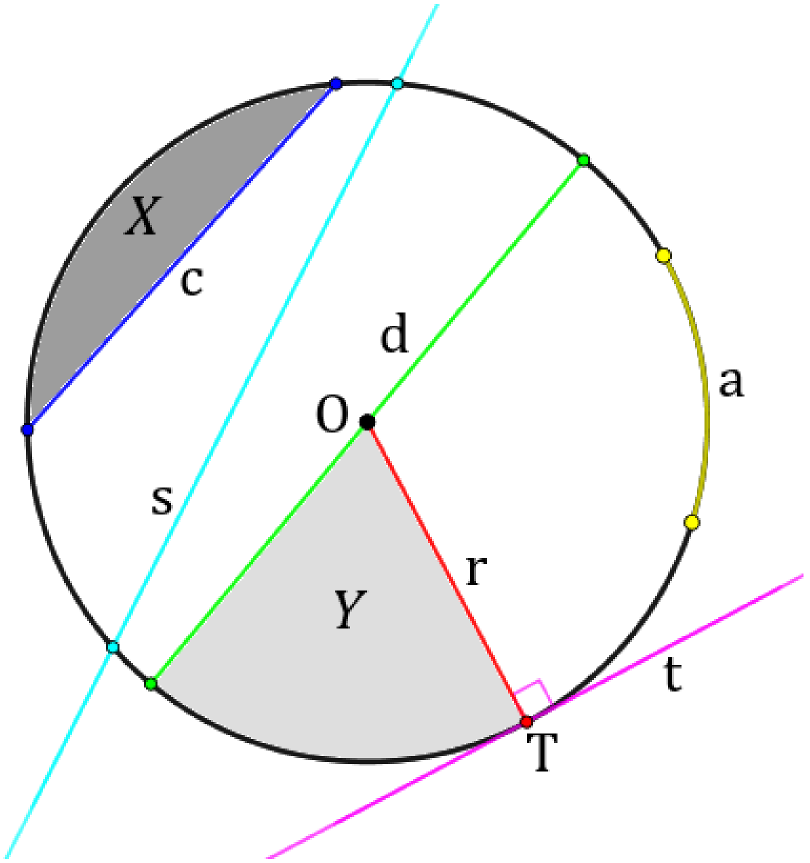
s – secant

a – arc

t – tangent line (T – tangent point)

X – segment

Y – sector



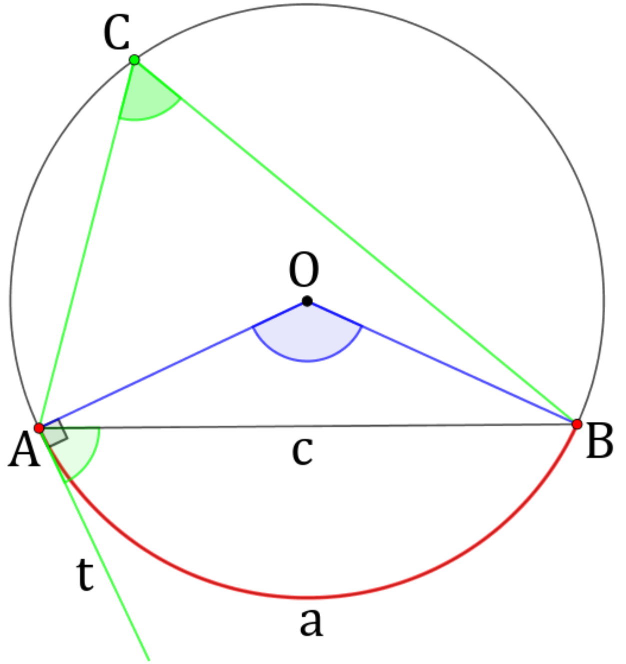
Additional related terms

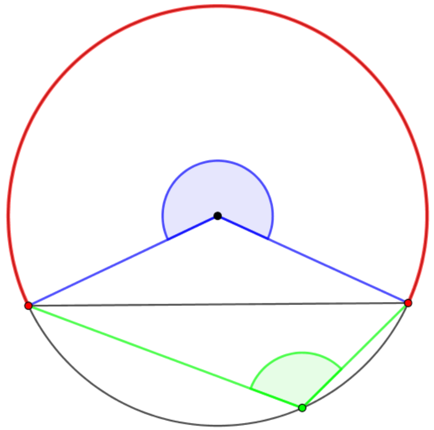
Central angle
The vertex of the angle is the center *and* the legs are lines of radii. E.g. ∢AOB

(Central angles subtended by arcs of the same length are equal.)

Inscribed angle
The vertex of the angle lies on the circle *and* legs are two secants, or one secant and one tangent line. E.g. ∢ACB

In other words, arc a (or chord c) subtends angle ∢ACB at point C.

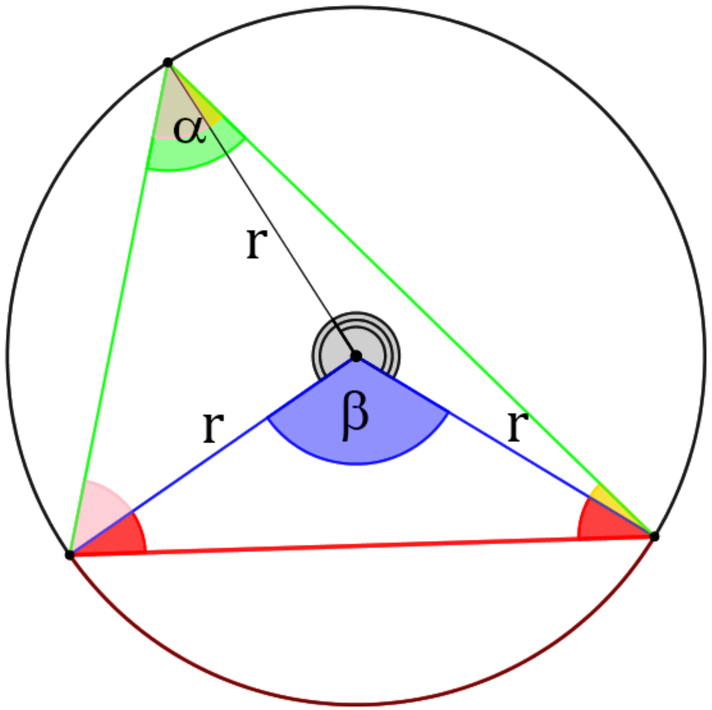


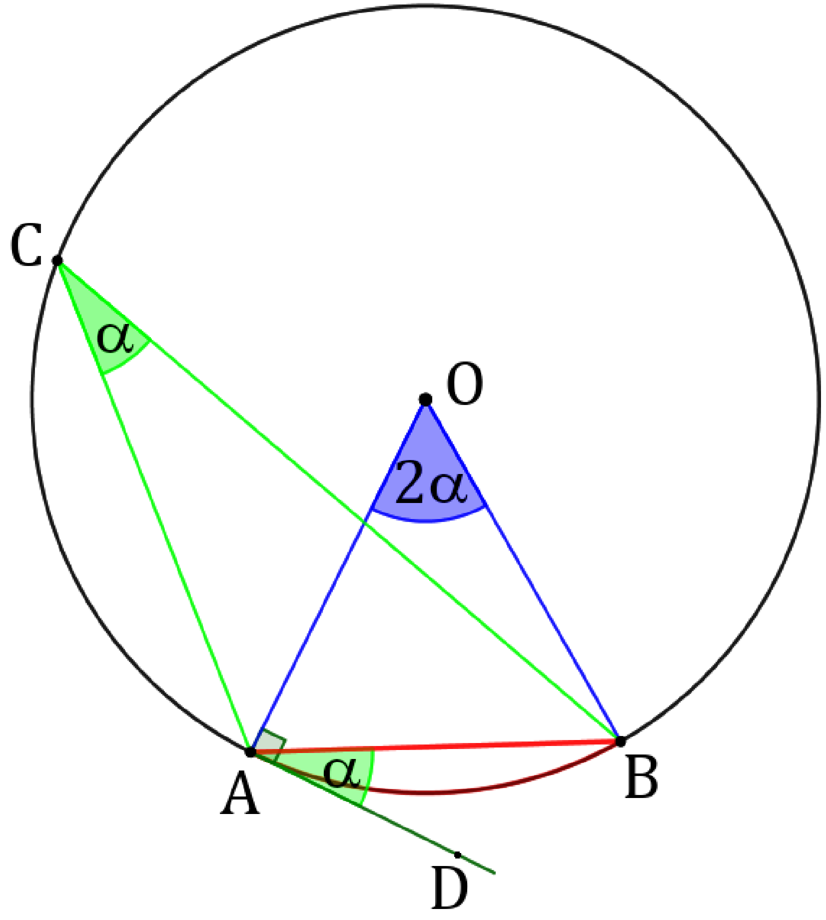


BASIC THEOREMS ABOUT CIRCLES – 1

If AB is an arc (of a circle) then the measure of central angle ∢AOB is twice the measure of inscribed angle ∢ACB.
(*Inscribed angle theorem*)

Sketch of the proof



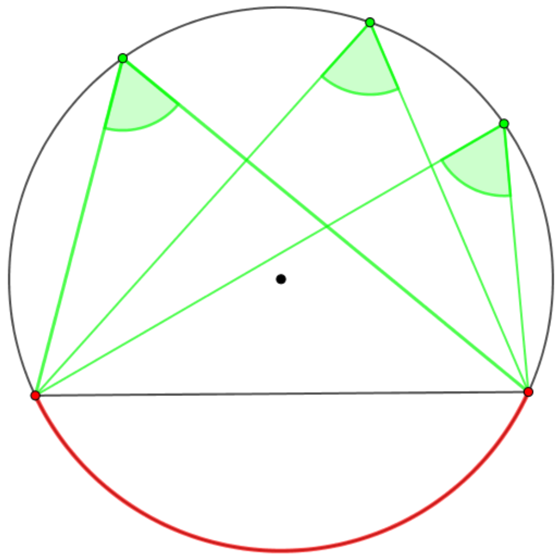


α=∢ACB= ∢DAB

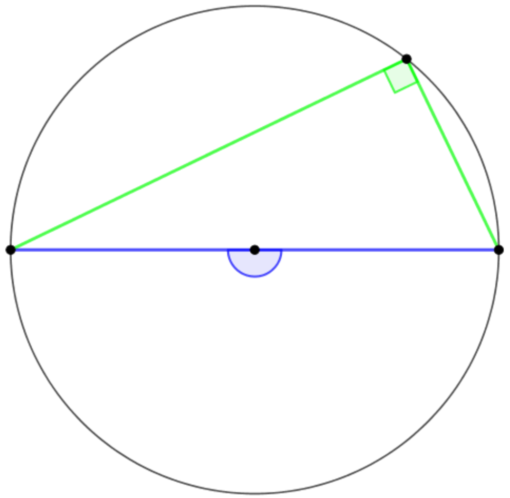
Basic theorems about circles – 2

Important corollaries

Any two inscribed angles (in a circle) with the same intercepted arc are congruent.



Thales' theorem
An inscribed angle in a semicircle is a right angle.



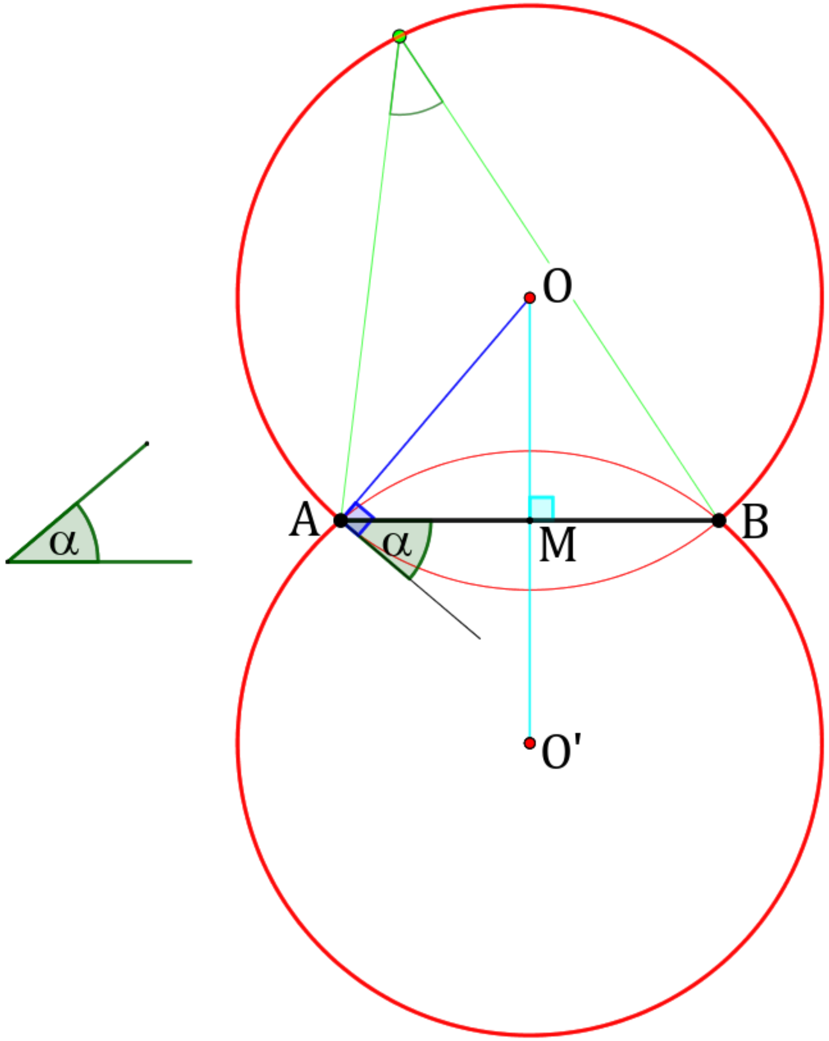
An essential construction – subtended angle

Let segment AB and angle α be given.

How can we find every point (vertex) where segment AB subtends angle α ?

⟶ finding the locus of the vertex

Locus (plural loci) is the set of points that satisfy the given condition.

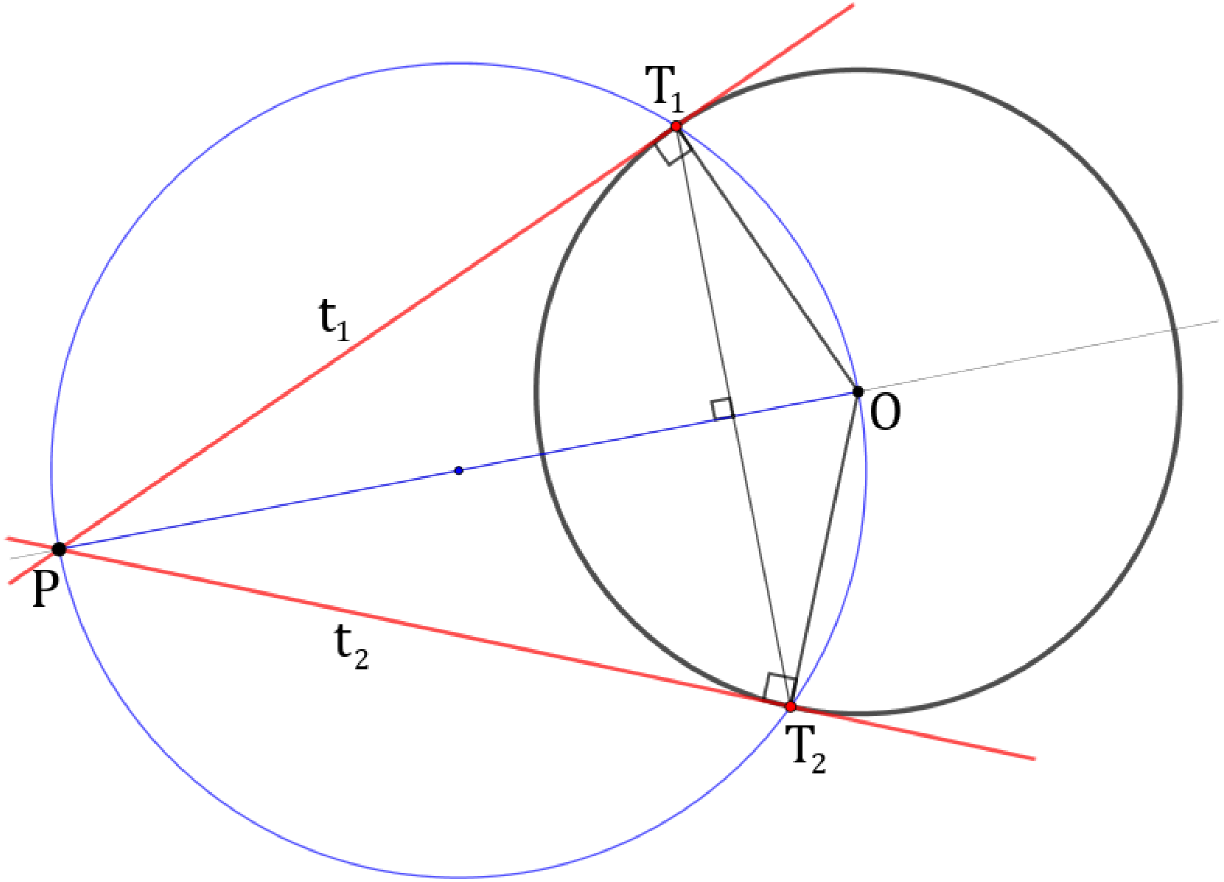


✍ lecture & practical

Construction of tangent lines to a circle

An arbitrary circle and an exterior point are given.

Construct tangent lines to the circle which pass through the given point.



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Construction of common tangent lines to circles – 1

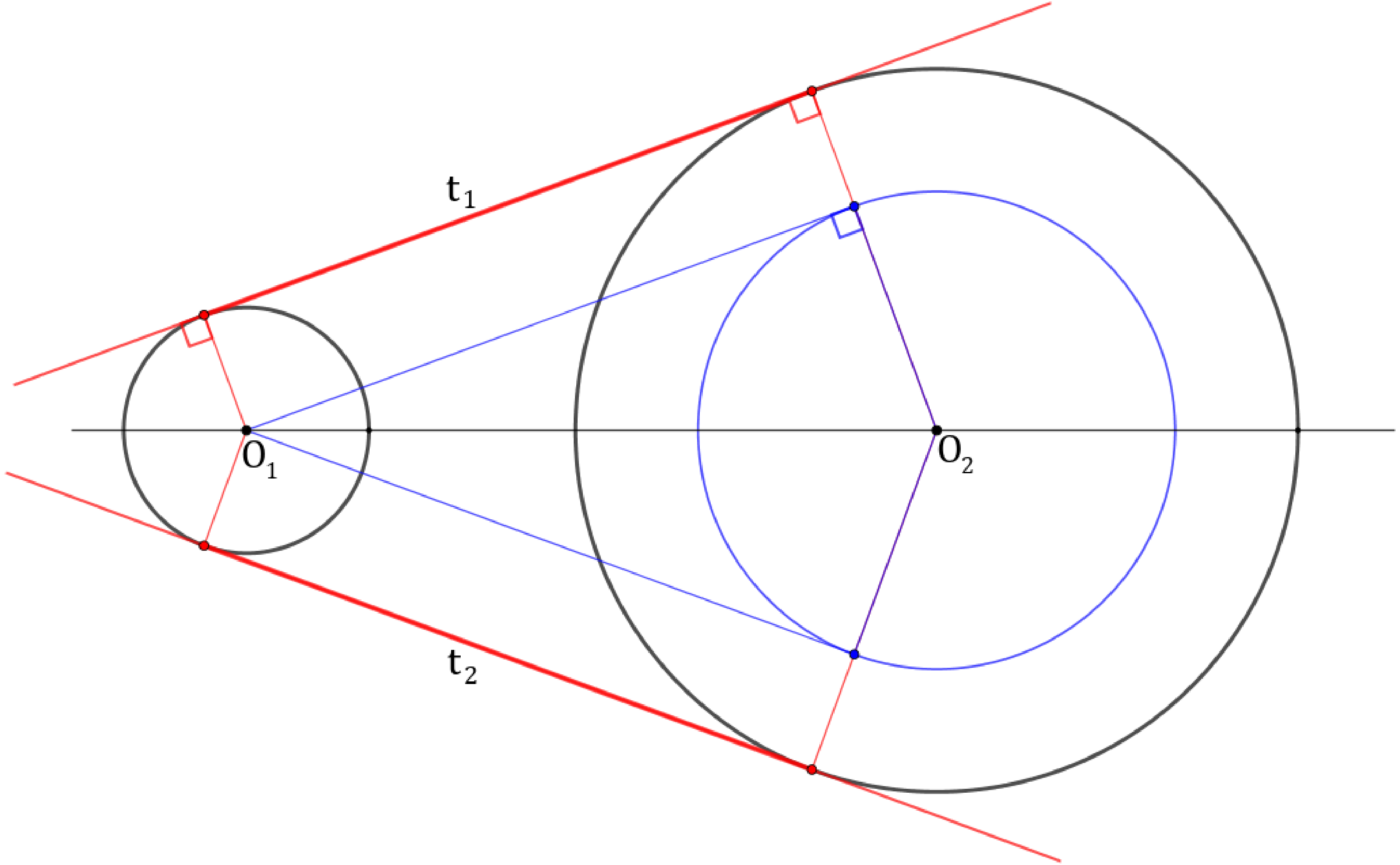
Two arbitrary disjoint circles are given. Construct all common tangent lines to the given circles.

Part 1 – *Outer tangents*

Circle 1 – center O_1, radius r_1

Circle 2 – center O_2, radius r_2

blue circle – center O_2, radius r_2-r_1



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Construction of common tangent lines to circles – 2

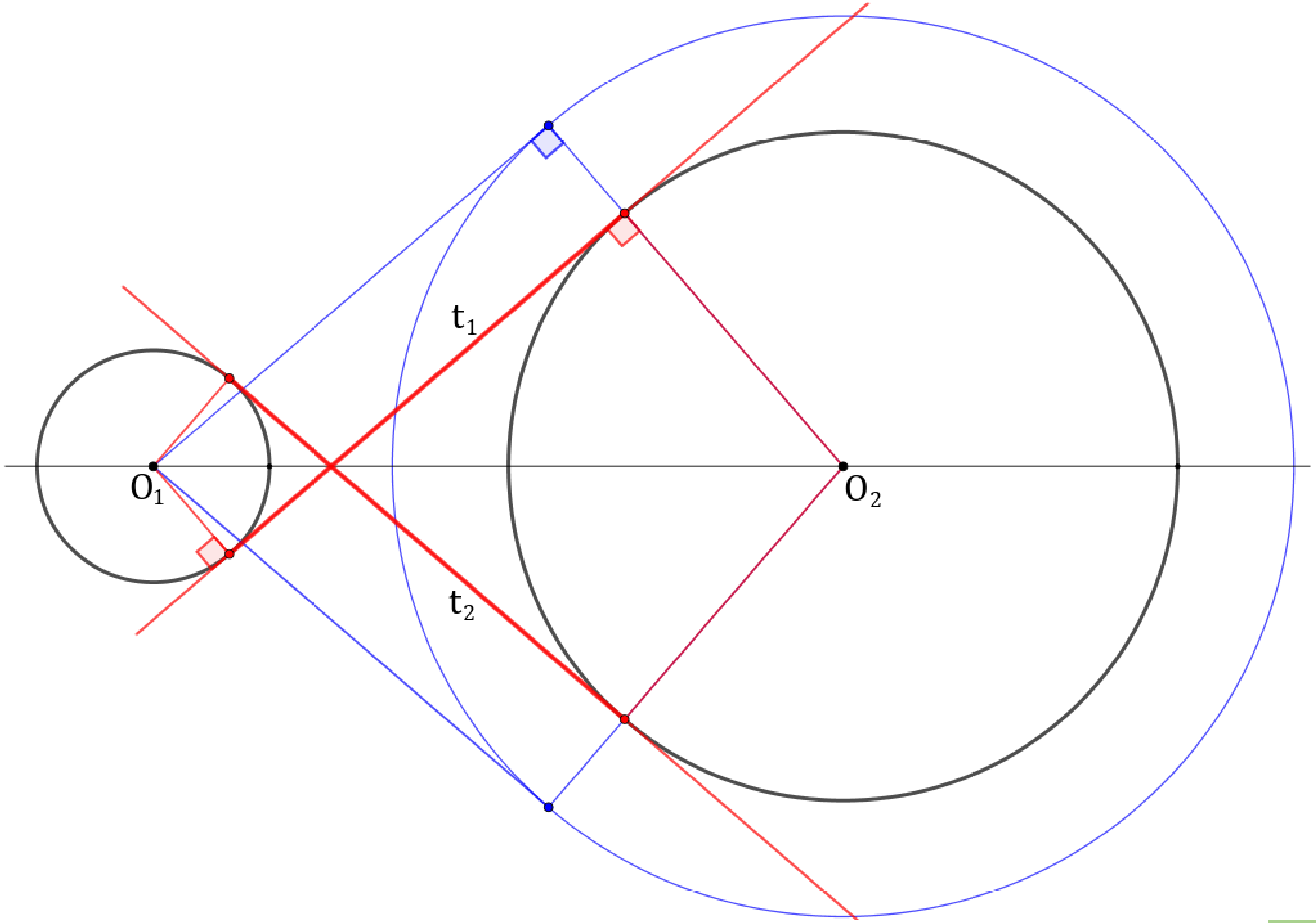
Two arbitrary disjoint circles are given. Construct all common tangent lines to the given circles.

Part 2 – *Inner tangents*

Circle 1 – center O_1, radius r_1

Circle 2 – center O_2, radius r_2

blue circle – center O_2, radius r_2+r_1



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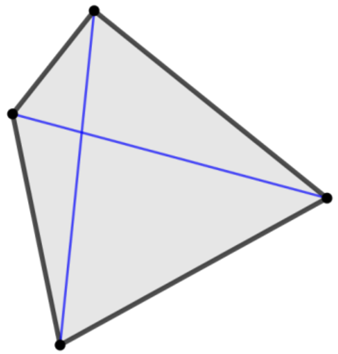
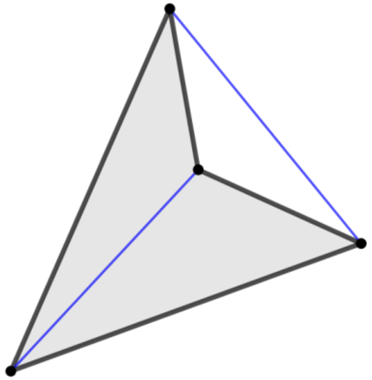
QUADRILATERALS

A *quadrilateral* has 4 sides and 4 vertices.

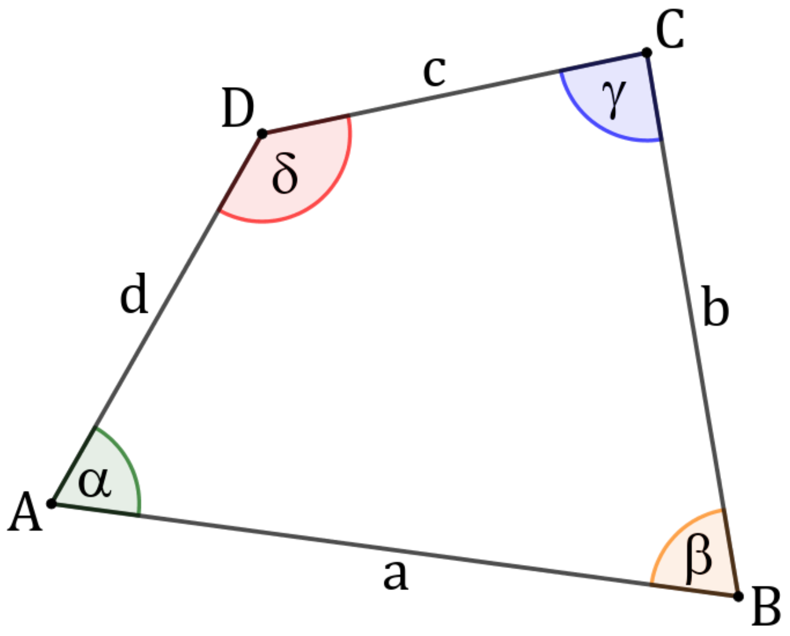




The *diagonals* of a quadrilateral connect two opposite vertices.

Convex (planar) quadrilateral:
every angle is acute, right or obtuse angles ($0° < \alpha < 180°$).

Concave (planar) quadrilateral:
exactly one angle is a reflex angle ($180° < \alpha < 360°$).
Its shape is like an arrowhead.



> The sum of the interior angles of a (planar) quadrilateral is 360°.

Hint of the proof: ABCD can be divided into two triangles by one of the diagonals.

Kites (deltoids)

A *kite* has two pairs of equal-length adjacent sides.

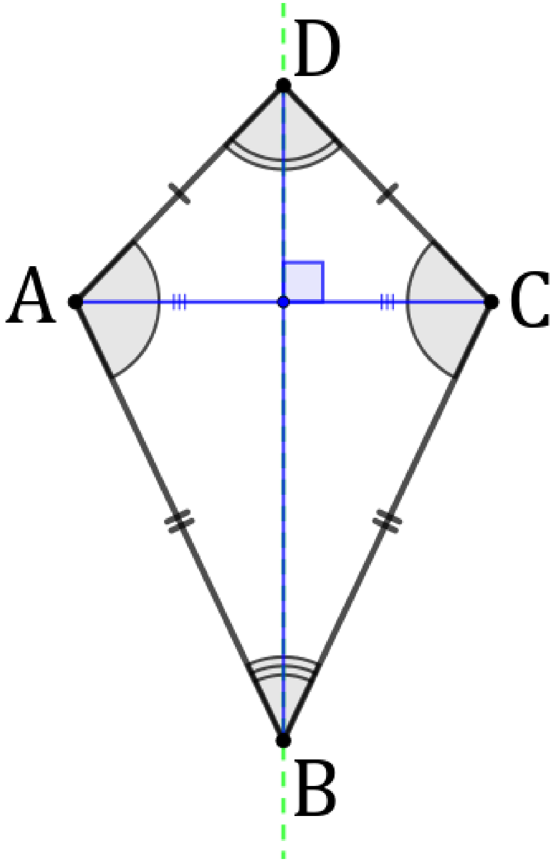
(E.g. AD and CD are adjacent to each other.)

A kite has an axis of symmetry.
(This axis is one of the two diagonals of the kite.)

Due to symmetry, the following statements are true:

One of the diagonals of a kite is the perpendicular bisector of the other diagonal.

The axis of symmetry of a kite is the angle bisector of two interior angles whose vertices lie on the axis.

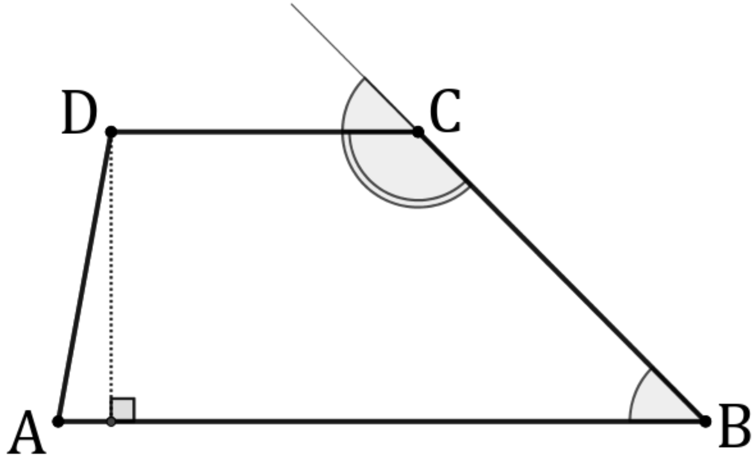


Remark: Convex and concave kites also exist.

Trapeziums (trapezoids) – 1

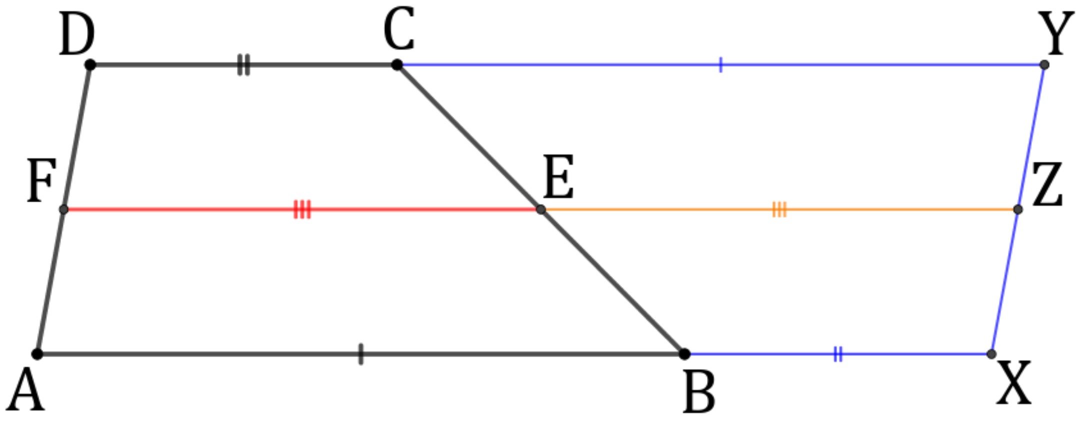
A *trapezium* is a convex quadrilateral
which has at least one pair of parallel sides.

The parallel sides are the *bases* of the trapezium,
the other two sides are called the *legs*.



The sum of two interior angles on the same leg is 180°.

The *midsegment* of a trapezium joins
the midpoints of the legs (and is parallel to bases).

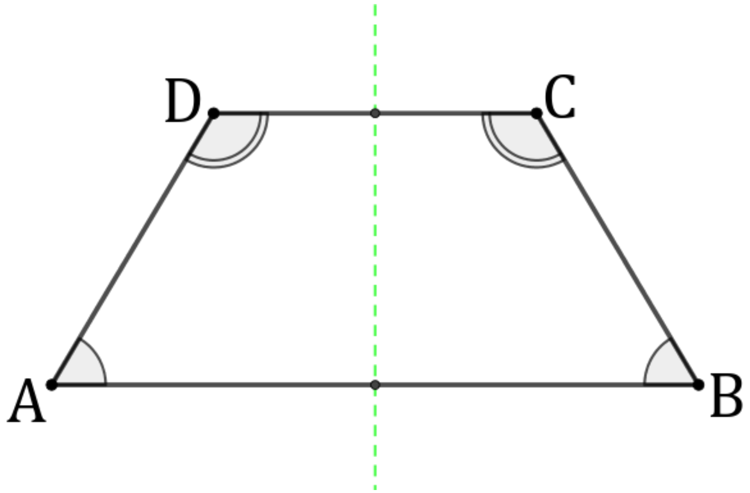


The length of the midsegment of a trapezium
is the arithmetical mean of the two bases.

Trapeziums (trapezoids) – 2

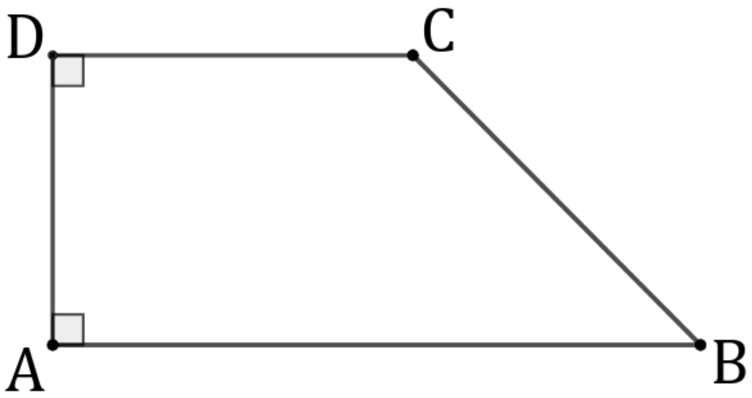
The legs of an *isosceles trapezium* have the same length.

Therefore, an isosceles trapezium has an axis of symmetry.



A *right trapezium* has two adjacent right angles.

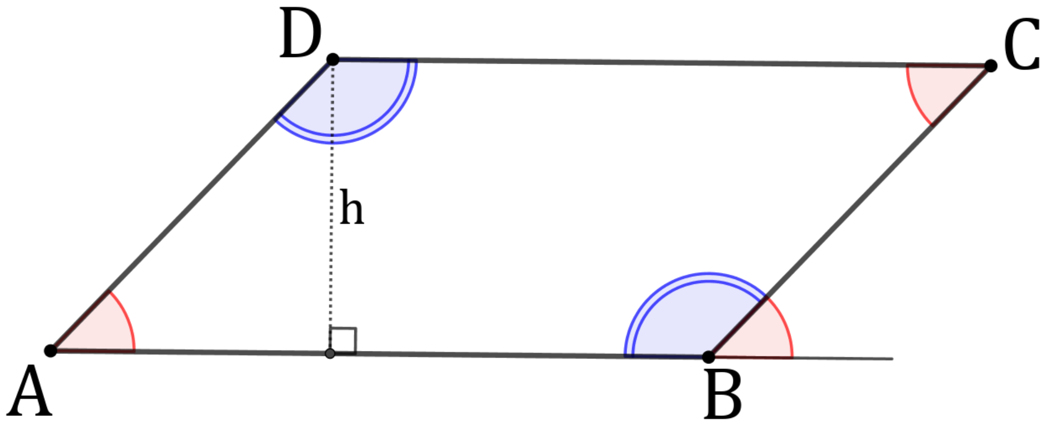
The right angles lie on the same leg.



Parallelograms

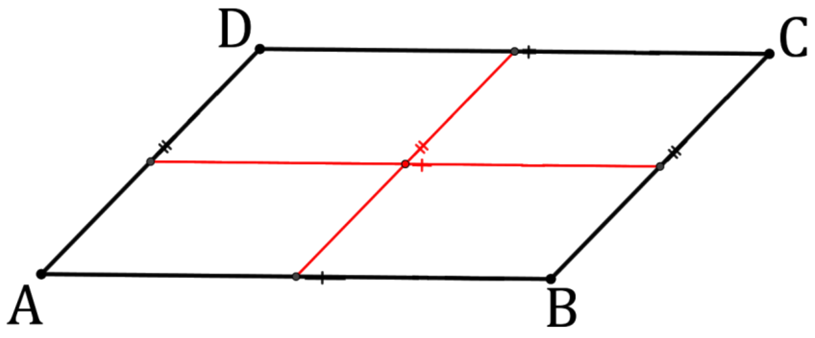
A *parallelogram* has two pairs of parallel sides.

$\Longrightarrow$ The opposite angles have the same measurement.

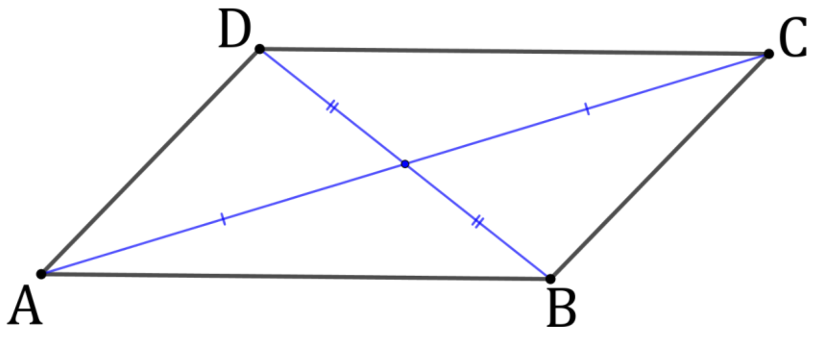


A *midsegment* of a parallelogram joins the midpoints of parallel sides.

A midsegment of a parallelogram is parallel to two sides and these segments have the same length.

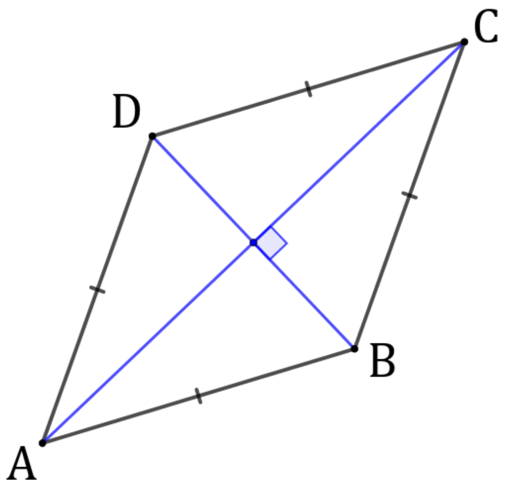


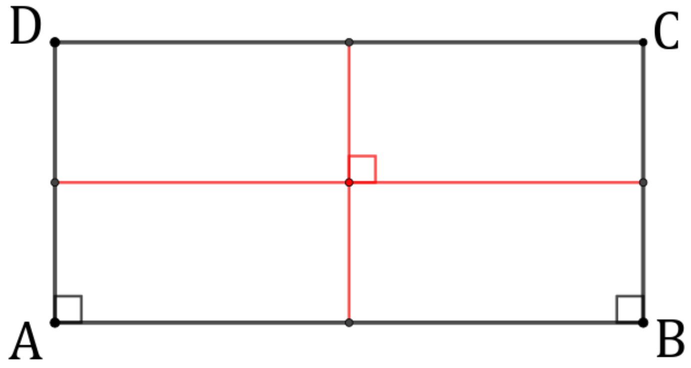
The diagonals of a parallelogram bisect each other.

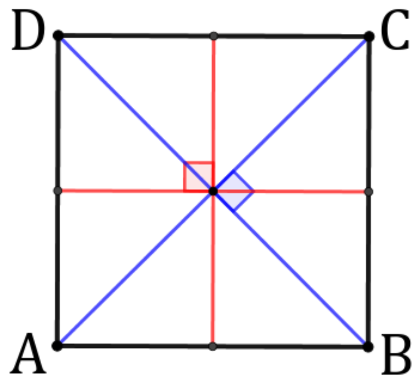


Rhombuses, rectangles, squares

Special cases







A *rhombus* is a parallelogram whose sides are equal.

A *rectangle* is a parallelogram whose angles are right angles.

A *square* is a rectangle whose sides are equal.

or

A square is a rhombus whose angles are right angles.

The diagonals of a rhombus are perpendicular to each other.

The midsegments of a rectangle are perpendicular to each other.

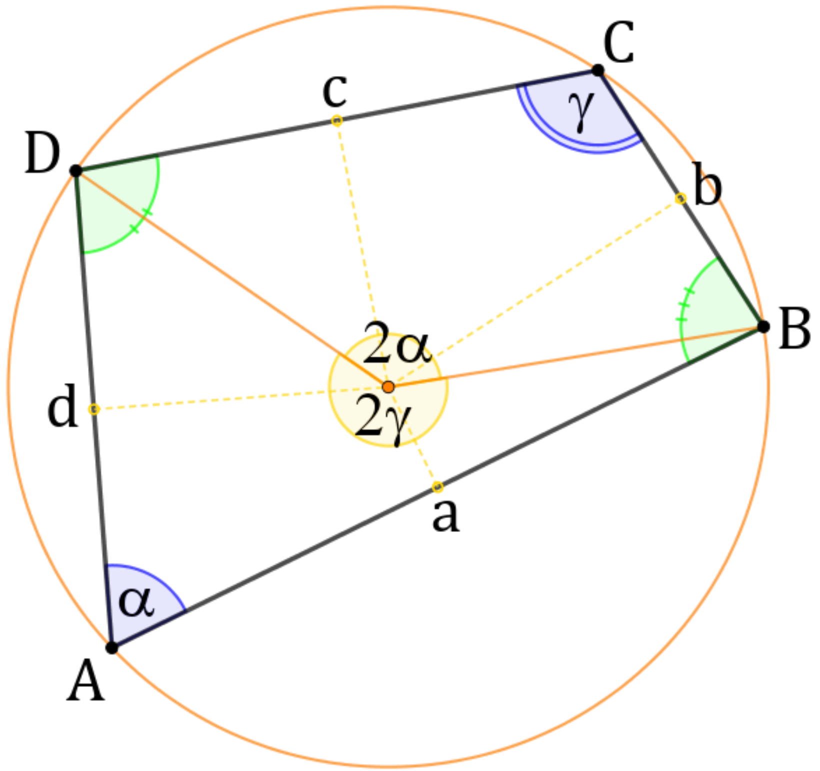
Cyclic quadrilaterals

A *cyclic quadrilateral* (or inscribed quadrilateral) has four sides which are chords of the same circle.

> A quadrilateral is a cyclic quadrilateral if and only if its opposite angles are supplementary.

The sum of *supplementary angles* is 180°.

Proof ⟶ the inscribed angle theorem



Remark: The perpendicular bisectors of a cyclic quadrilateral meet at the center of its circumscribed circle.

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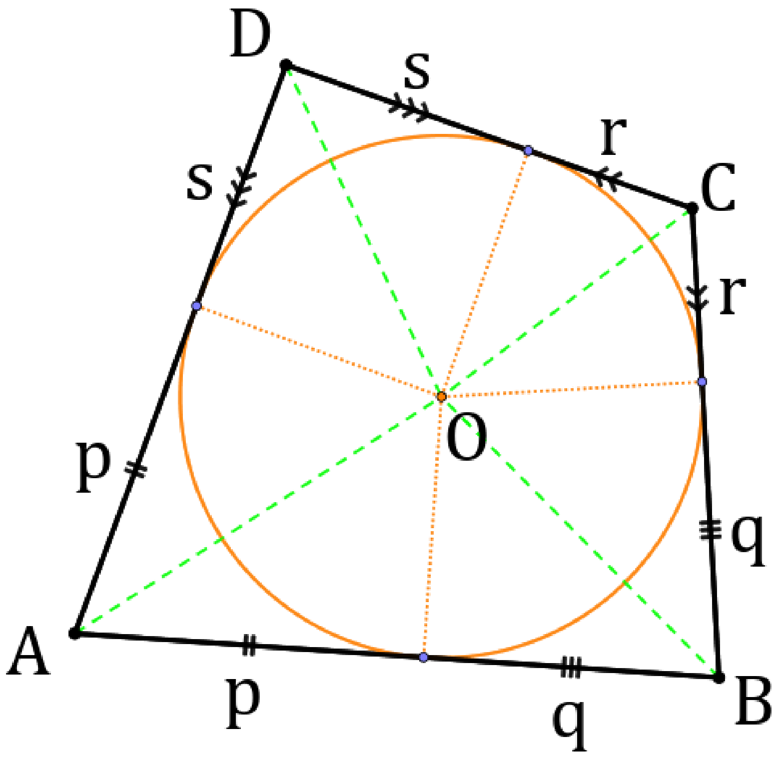
Tangential quadrilaterals

A *tangential quadrilateral* (or circumscribed quadrilateral) has four sides – as lines – which are tangent lines of the same circle.



A quadrilateral is a tangential quadrilateral if and only if the two sums of the lengths of the opposite sides are equal.

Proof ⟶ construction of tangent lines to a circle

Remark: The angle bisectors of a tangential quadrilateral meet at the center of its inscribed circle.

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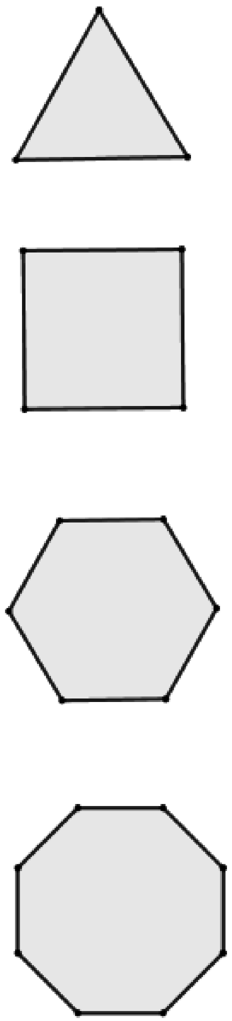
REGULAR POLYGONS

A *regular polygon* is an n-sided polygon in which the sides are all the same length and are symmetrically placed about a common center.

A regular polygon is equilateral and equiangular.



Some regular polygons are "constructible" using the compass and straightedge.
For example, if n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30, 32, ...

(For reason see this link ⟶ geometry & number theory)

Well-known constructions:

n = 3 (equilateral triangle), n = 4 (square),
n = 6 (regular hexagon), n = 8 (regular octagon)

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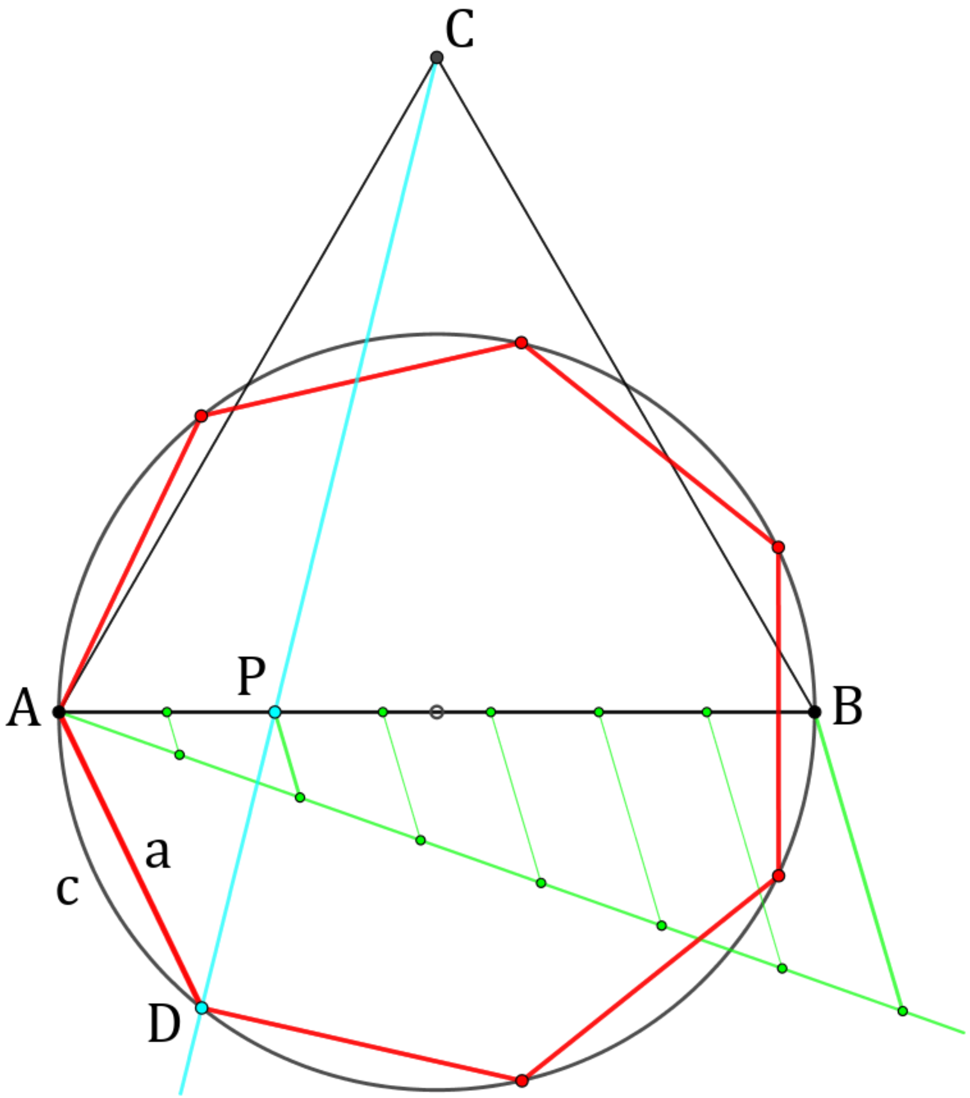
An approximate construction of regular polygons

For example, n = 7 (regular heptagon)

Sketch of steps of the construction:

1. △ABC is an equilateral triangle
2. side AB as a diameter of a circle (denoted by c)
3. drawing circle c
4. dividing AB into n(=7) equal parts (see here)
5. choosing the second point of this division (denoted by P)
6. drawing ray CP
7. the intersection point of ray CP and circle c is point D
8. segment AD is the *approximate* length of the inscribed regular polygon



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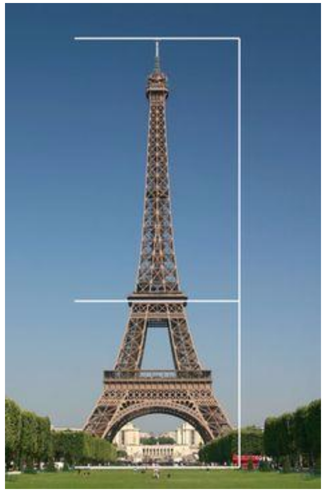
THE GOLDEN RATIO

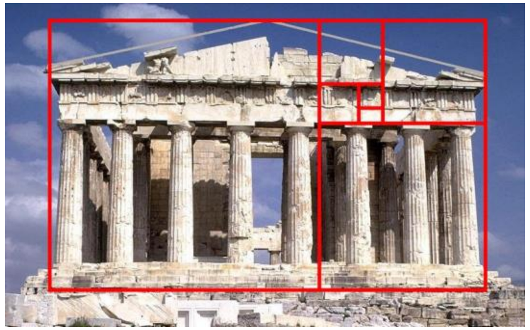
Numbers a and b (a>b) are in *the golden ratio* if the ratio of a to b is the same as the ratio of a+b to a:

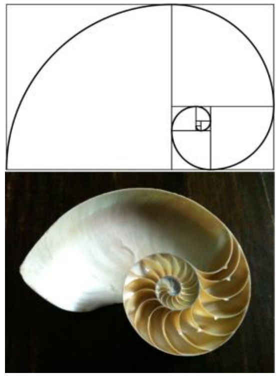
$$\frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{a}+\mathrm{b}}{\mathrm{a}} = \varphi$$

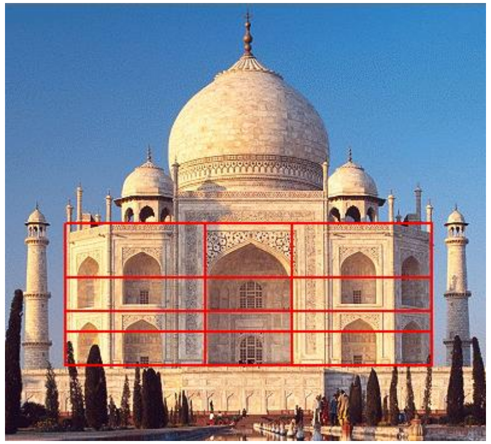
$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

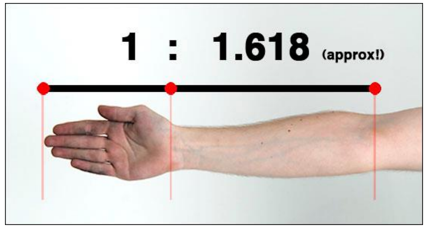
Golden ratio in architecture







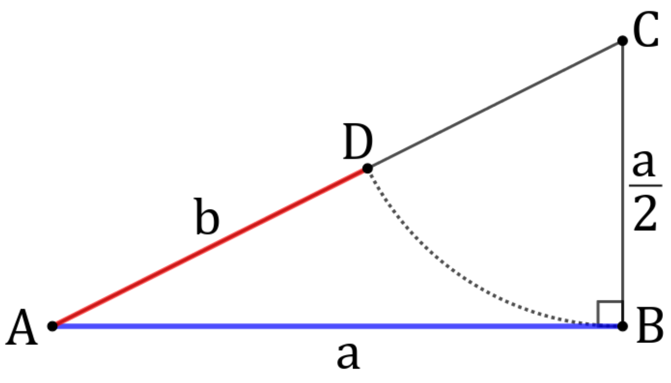




Sources: Internet

Constructions on the golden ratio

If we know the length of segment a:
using a special right triangle



1. segment AB=a
2. drawing segment $BC = \frac{a}{2}$ which is perpendicular to a
3. segment $AC = b + \frac{a}{2}$
4. BC=DC $\Longrightarrow$ AD=b

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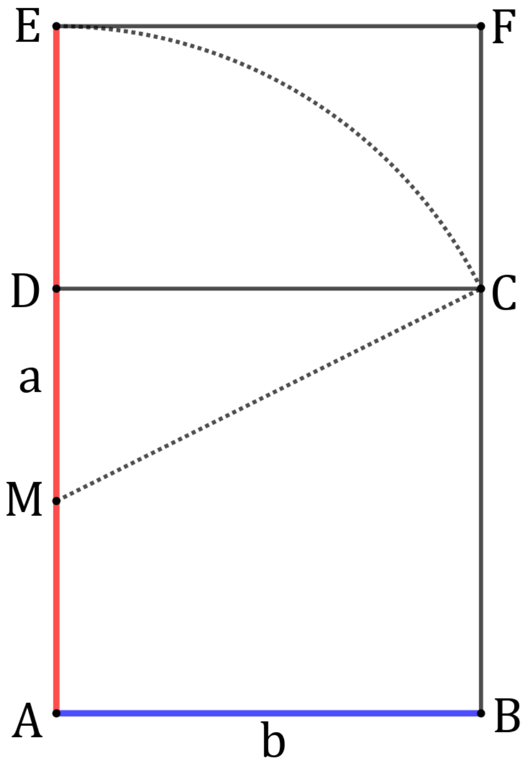
If we know the length of segment b:
constructing *the golden rectangle*

1. segment AB=b
2. drawing square ABCD
3. M is the midpoint of segment AD
4. drawing a circle whose center is M and radius is MC
5. this circle intersects ray AD at point E
6. AE=a



Moreover, b and DE are in the golden ratio as well.

Connection between the golden ratio and regular polygons

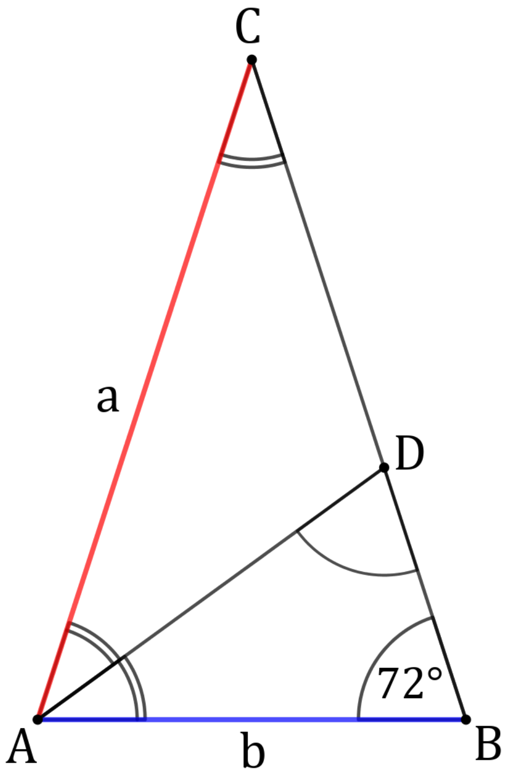
Golden triangle

AB=b

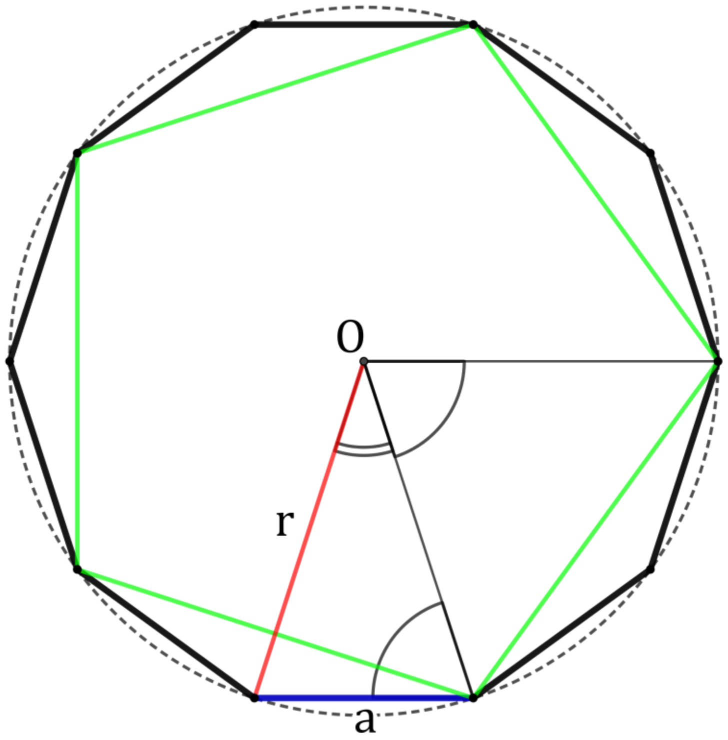
AC=BC=a

∢ABC=∢BAC=72°

∢ACB=36°



In the case of regular decagon, its side a and the radius of the circumscribed circle r are in the golden ratio.

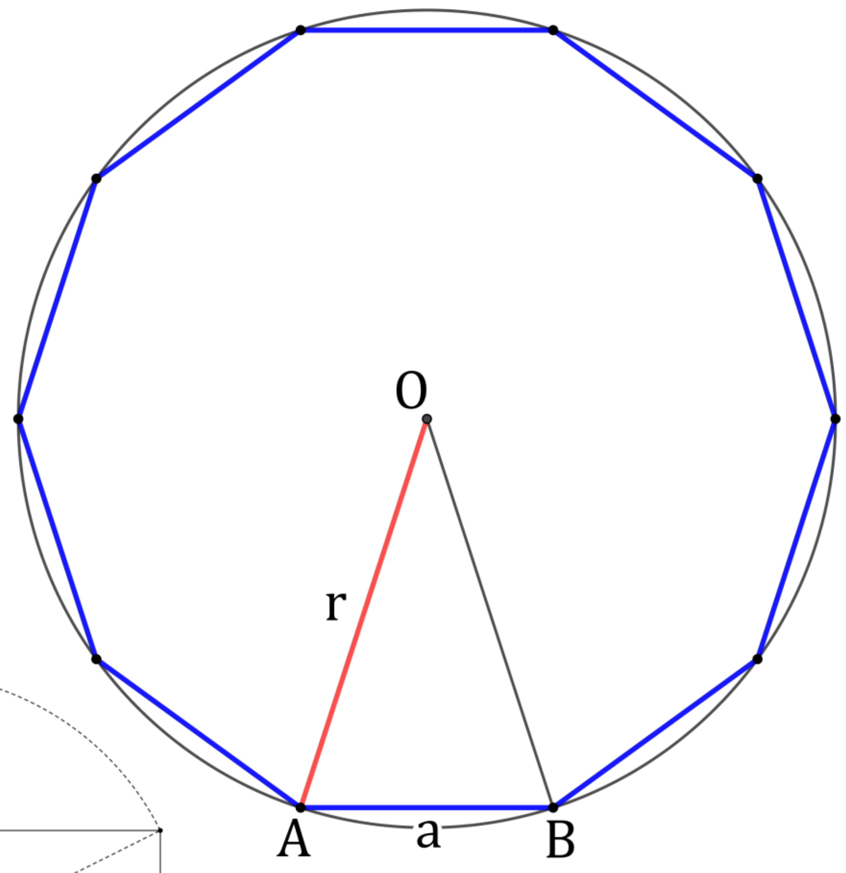


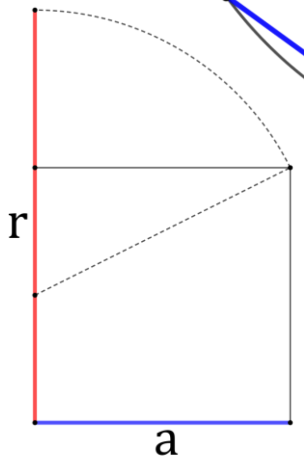
⟹ We can construct regular decagon (and regular pentagon).

Constructions on regular decagon

Let segment AB=a be given. Construct a regular decagon whose side is the given segment.

1. constructing the bigger part of the golden ratio using the golden rectangle $\Longrightarrow$ radius r
2. drawing an arc of radius r and center A
   and drawing an arc of radius r and center B
3. the intersection point of arcs is the center of the circumscribed circle (point O)
4. drawing the circumscribed circle
5. measuring 9 times segment AB=a





Remark: Every second vertex forms a regular pentagon.

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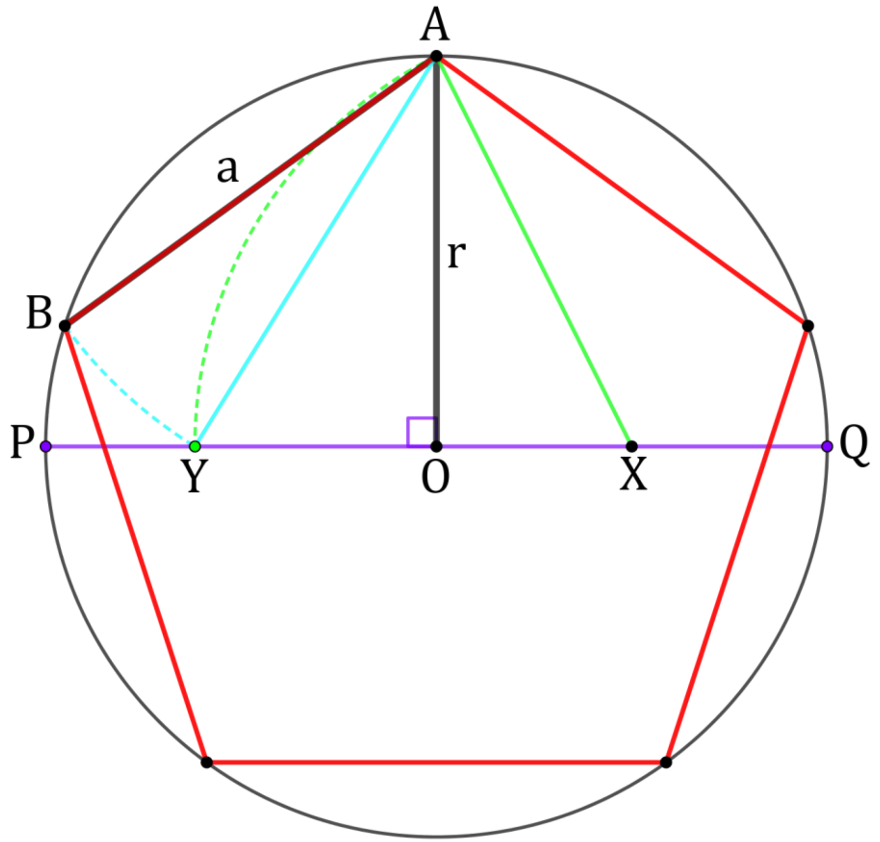
Constructions on regular pentagon – 1

Using a regular decagon or... Construction on regular pentagon if...

the radius of the circumscribed circle is given.

1. segment OA=r (as the radius) and the circumscribed circle
2. drawing the perpendicular diameter PQ to r
3. X is the midpoint of OQ
4. drawing a circle of radius XA and center X
5. Y is the intersection point of PQ and the previous circle
6. drawing a circle of radius AY(=a) and center A
7. this circle intersects the circumscribed circle at point B where B is a vertex of the regular pentagon
8. drawing missing vertices (using the length of AB=a)



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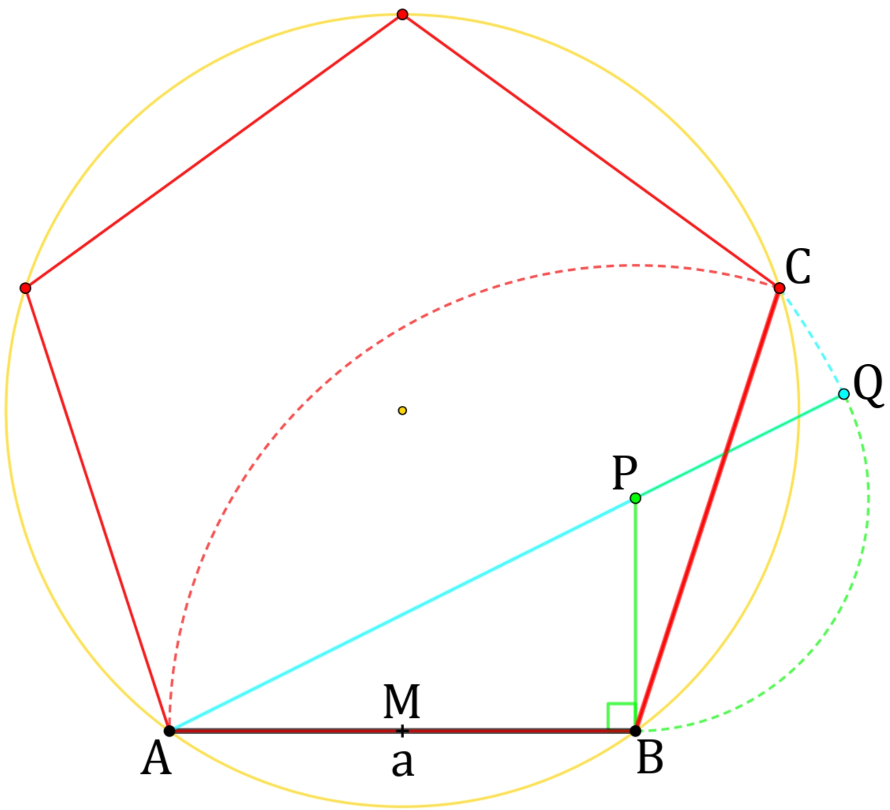
Constructions on regular pentagon – 2

Construction on regular pentagon if the side of the pentagon is given.

1. segment AB=a (as the side)
2. drawing a perpendicular segment at point B to AB whose length is $\frac{a}{2}$ (its endpoint is P)
3. ray AP
4. PB=PQ (using a circle) $\Longrightarrow$ point Q (lies on ray AP)
5. drawing a circle of radius AQ and center A
6. drawing a circle with of radius a and center B
7. these circles intersect each other at point C (as a new vertex)
8. construct missing vertices (using the circumscribed circle of triangle $\triangle$ABC)



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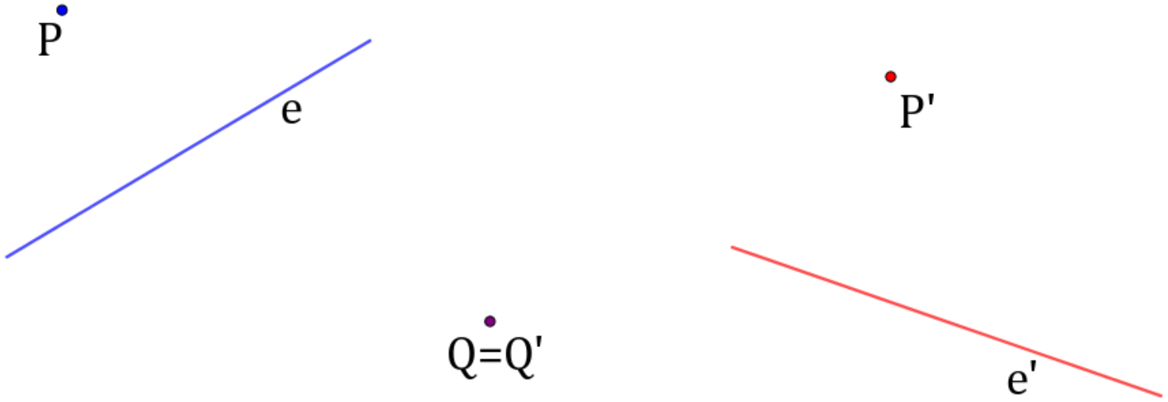
TRANSFORMATIONS

A **geometric transformation** is a one-to-one correspondence between two sets of points.

The *image point* of a point P is denoted by P'.



Fixed point: Q=Q'
Pointwise fixed line:
every point of it is fixed point

Invariant line: l=l' as a set of points

Remarks:
– We can "combine" transformations. This act is called composing.
– The "reverse" transformation of a transformation is called *inverse* transformation.

ISOMETRIES (of a plane)

A **plane isometry** (or congruence) – denoted by $\cong$ – is a geometric transformation which preserves

- collinearity (the image of a line is a line),
- parallelism,
- distance,
- angle measure.

Direct/rigid/orientation-preserving isometry:
triangle $\triangle ABC$ and its image triangle $\triangle A'B'C'$ have the same orientation (both of them are clockwise or counter-clockwise).
$\longrightarrow$ translation, rotation

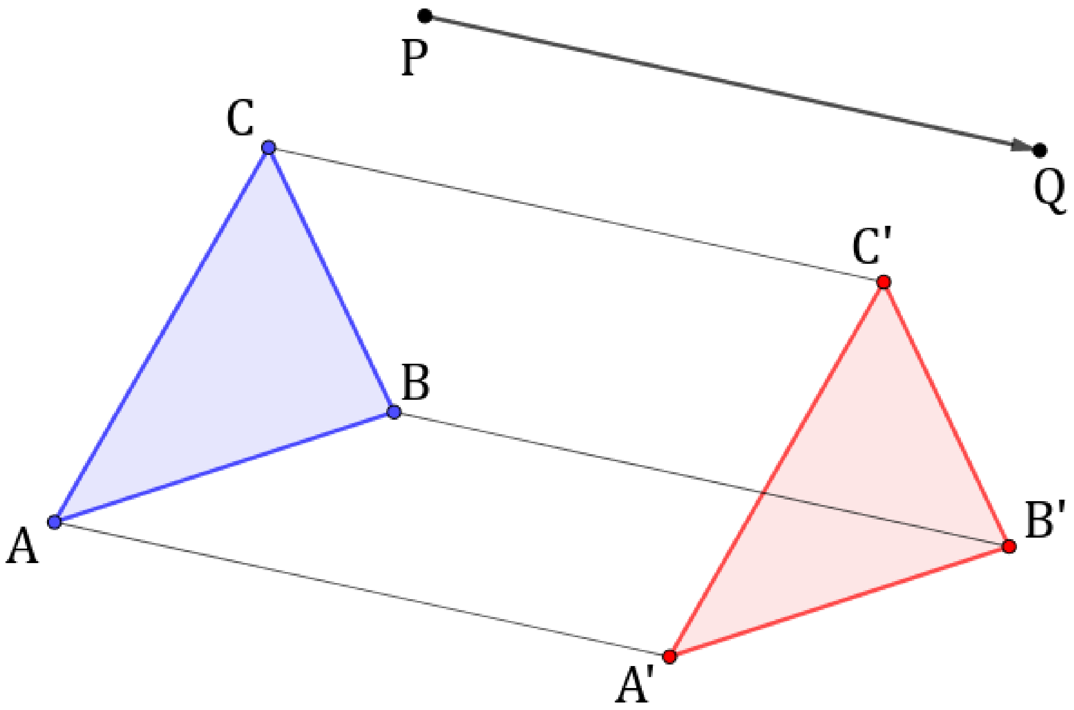
Indirect/opposite/orientation-reversing isometry:
triangle $\triangle ABC$ and its image triangle $\triangle A'B'C'$ have different orientations.
$\longrightarrow$ reflection

Translation

Translation is an orientation-preserving plane isometry which moves every point by the same distance in a given direction.

Translation is a geometric transformation with no fixed points.

The given distance and direction can be represented by a vector called *translation vector*.



Rotation

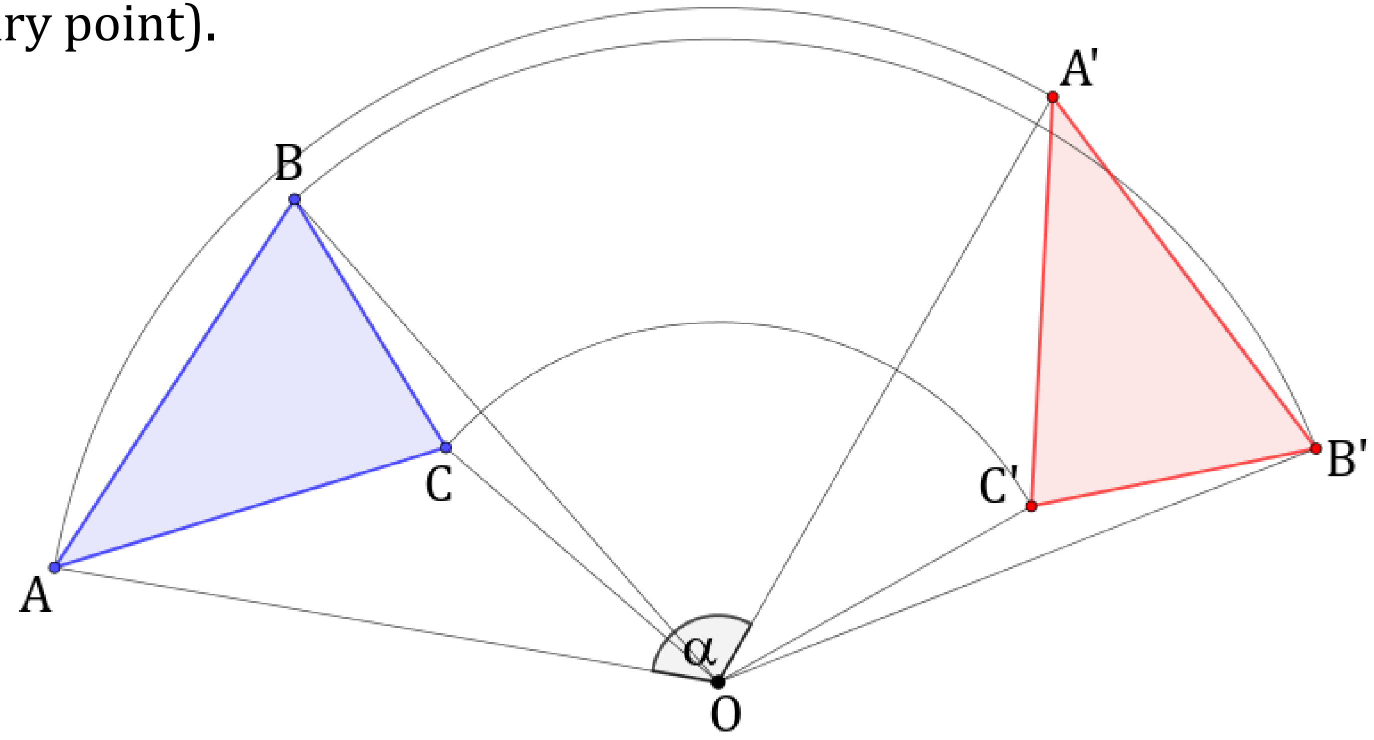
Rotation is an orientation-preserving plane isometry which has a fixed point (*center* O) and a given angle α so that the measure of any angle POP’∢ is α and OP=OP’ (where P is an arbitrary point).

The rotation is *clockwise* if $\alpha<0$.

The rotation is *counter-clockwise* if $\alpha>0$.

A shape is *rotationally symmetric* if
it looks the same after a particular rotation.

E.g. regular polygons

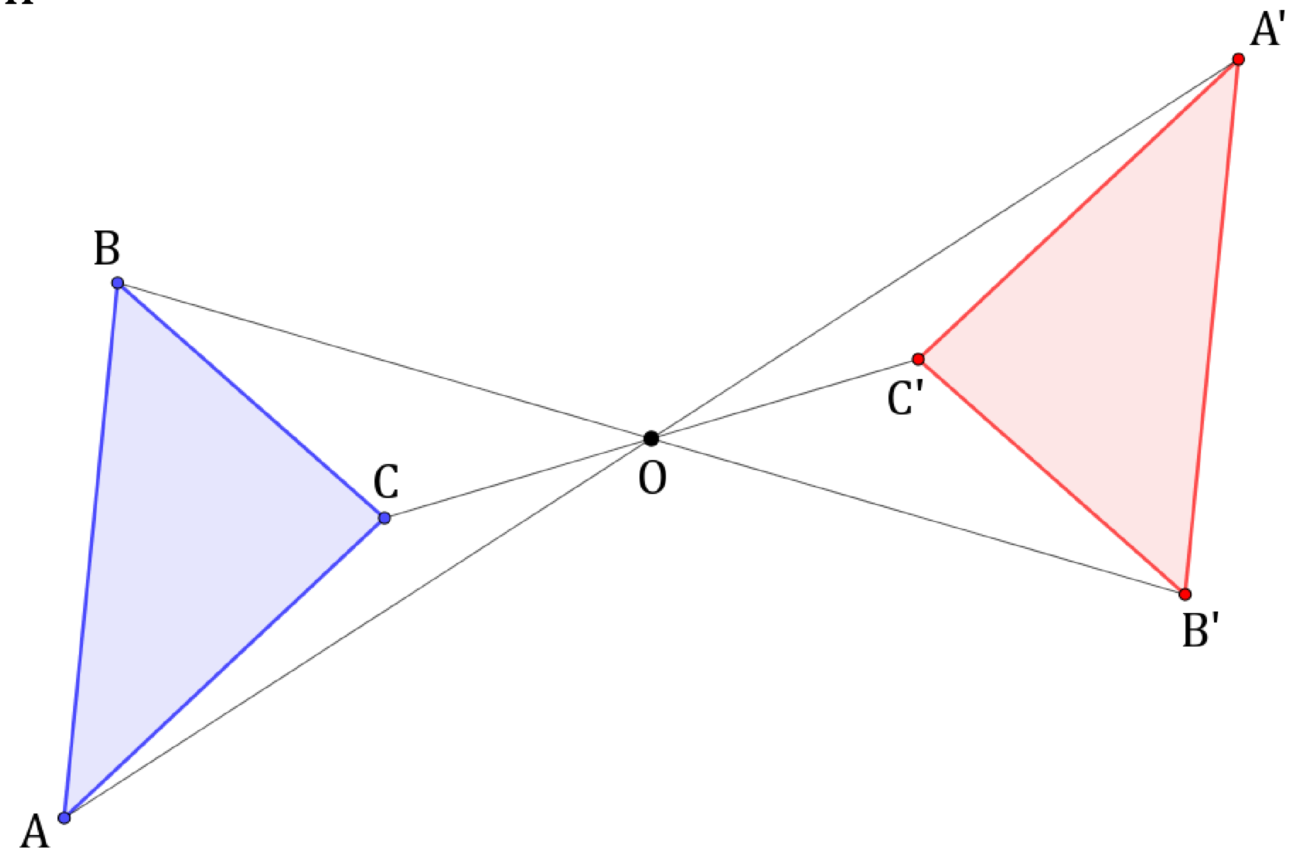


Trivial case: a rotation is **the identity** if the given angle is the zero (or $n\cdot 360°$, $n\in\mathbb{Z}$) angle.

A special rotation: **Point reflection**

Point/central reflection is a special rotation whose given angle is 180°.

Every line passing through the center is an invariant line of the point reflection.



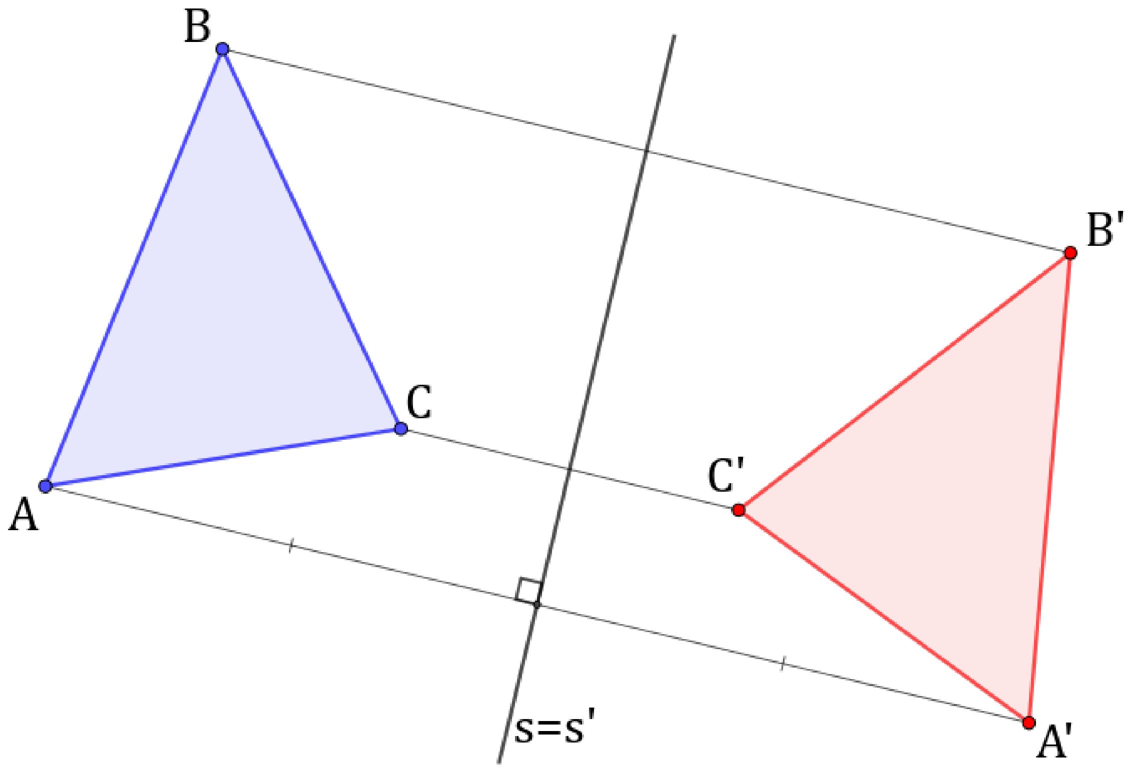
A shape is *centrally symmetric* if it looks the same after a point reflection.

E.g. parallelogram, regular polygons
(with respect to their centers)

Reflection

Reflection (or axial reflection) is an orientation-reversing plane isometry which has a pointwise fixed line (the *axis* of reflection) so that the axis is the perpendicular bisector of every segment PP’ (where P is an arbitrary point).

Every line PP’ is an invariant line of the reflection.



A shape is *reflection/line/mirror symmetric* if it looks the same after a reflection.
The axis is called the axis of symmetry.

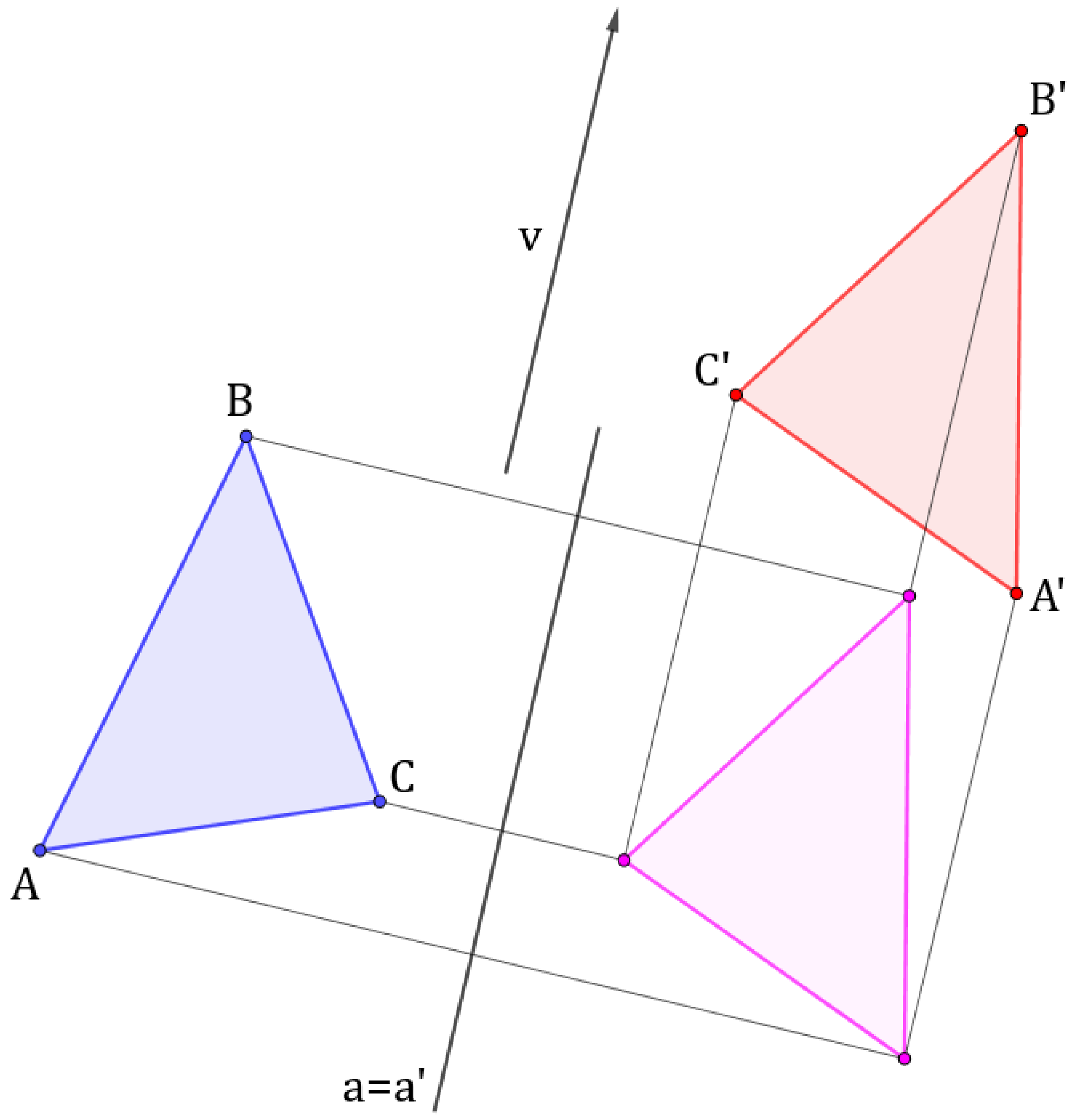
E.g. deltoid, rectangle, regular polygons

Applications of isometries 1.

An important composition of two isometries:

a reflection and a translation whose vector is parallel to the axis of the reflection

$\Longrightarrow$ *glide reflection*



Applications of isometries 2.

Some exercises

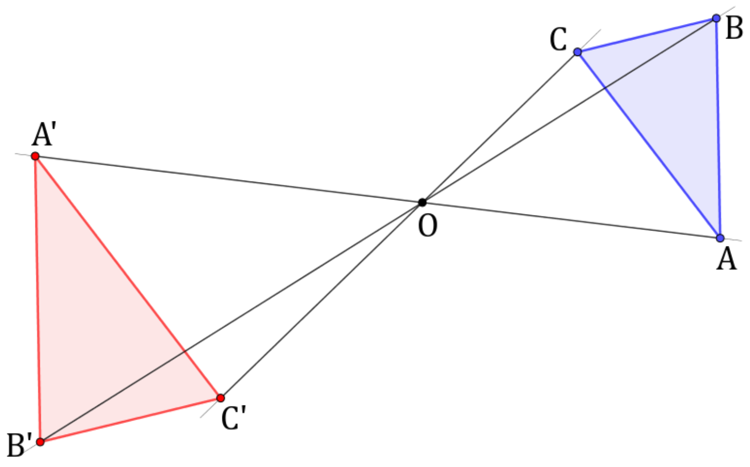
1. An arbitrary circle and a segment (whose length is smaller than the diameter of the circle) are given. Construct chords of the circle which are parallel and have the same length with the given segment. (Hint: translations)
2. The center of a square, and line l and k are given. One of vertices of the square lies on line l and another vertex lies on line k. Construct the square. (Hint: rotation)
3. A circle and a disjoint line l are given. Choose an arbitrary point P on the circle. Construct a line through point P so that the chord lying on this line is equal to the segment of P and the intersection point of this line and l. (Hint: point reflection)
4. There are two cities near the same bank of a river. Find the right place of a water disinfection station (on this bank) for both cities so that the distances of the cities from the station are minimal. (Hint: reflection)

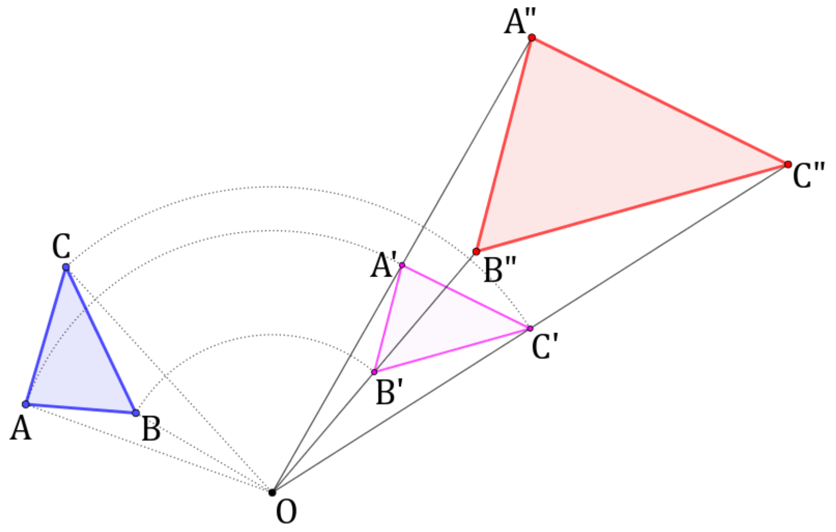
lecture

SIMILARITIES

A **plane similarity transformation** – denoted by $\sim$ – is a geometric transformation where the ratio of a segment and its image is constant, and it preserves

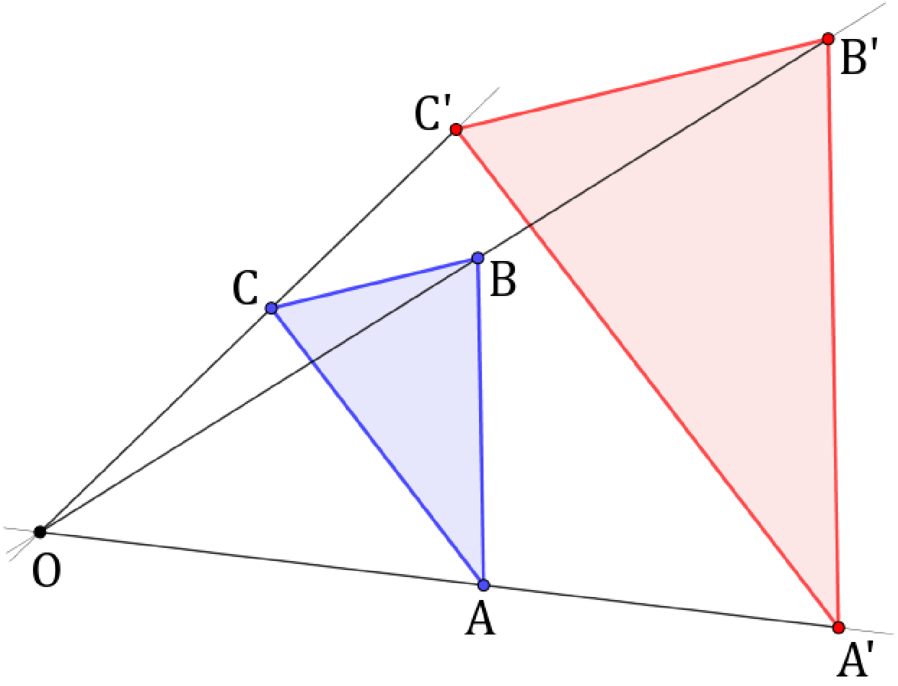
- collinearity,
- parallelism,
- angle measure.





Homothety 1.

Homothety (or homothecy or homogeneous dilatation) is a similarity transformation with a fixed point O and a real number λ so that for every point P its image point P' lies on line OP, and OP'=λ·OP .



homothetic center – point O
scale factor – real number λ (where $\lambda \neq 0$)

Every segment and its image are parallel.

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The scale factor of the inverse transformation is $\frac{1}{\lambda}$.

If A is the area of an object, then the area of the image object is $\lambda^2 \cdot A$.

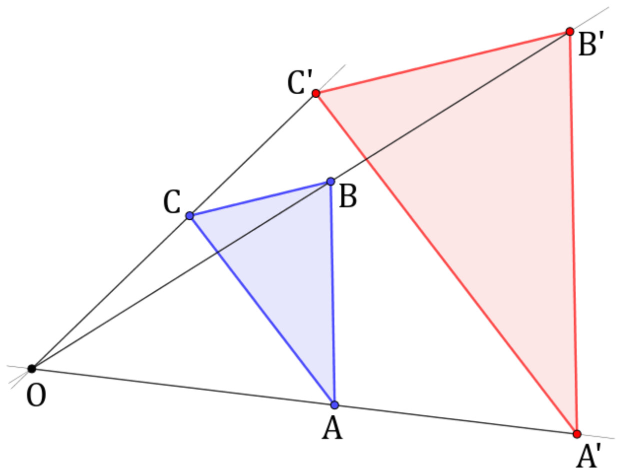
(In space, if V is the volume of an object, then the volume of the image object is $\lambda^3 \cdot V$.)

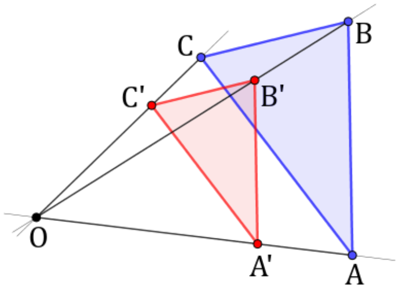
Homothety 2.

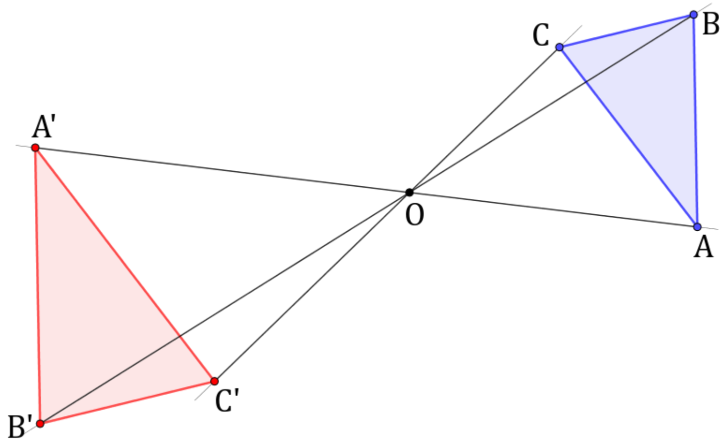
If $\lambda > 1$, then the object is enlarged by the scale factor.

If $0 < \lambda < 1$, then the object is shrunk by the scale factor.

If $\lambda < 0$, then the object is enlarged/shrunk and reflected by point O.





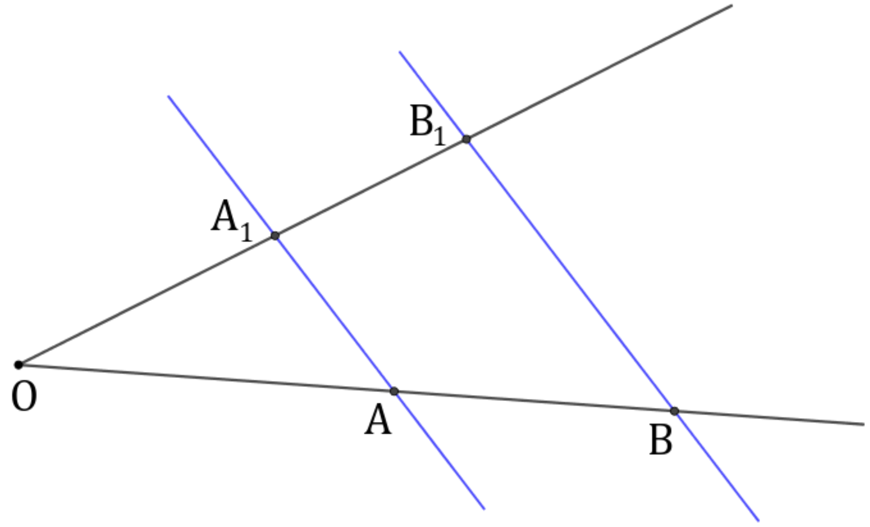


Intercept theorem

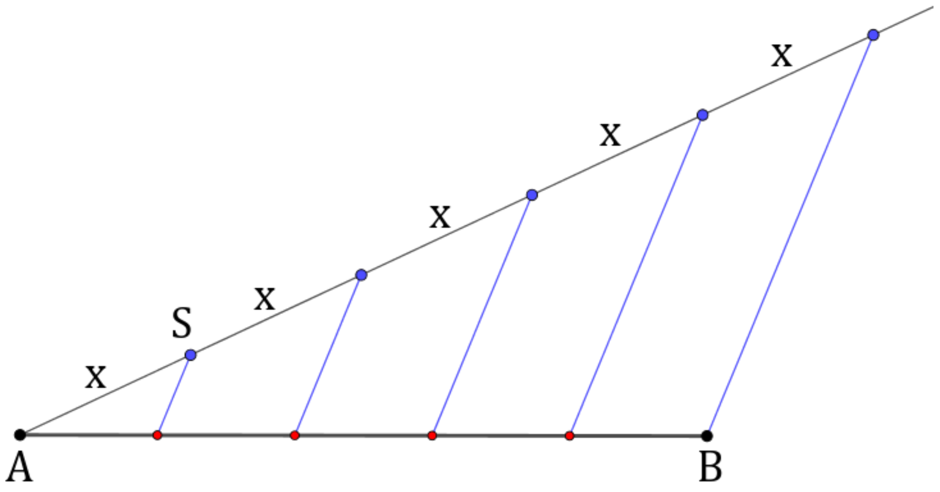
Intercept theorem: If two lines intersecting in point O are cut by parallel segments AA_1 and BB_1, then the following formulas are true:

$$\frac{OA}{OB} = \frac{OA_1}{OB_1} = \frac{AA_1}{BB_1}$$

$$\frac{OA}{AB} = \frac{OA_1}{A_1B_1}$$



Using this theorem, we can divide
an arbitrary segment AB into n equal parts:



lecture

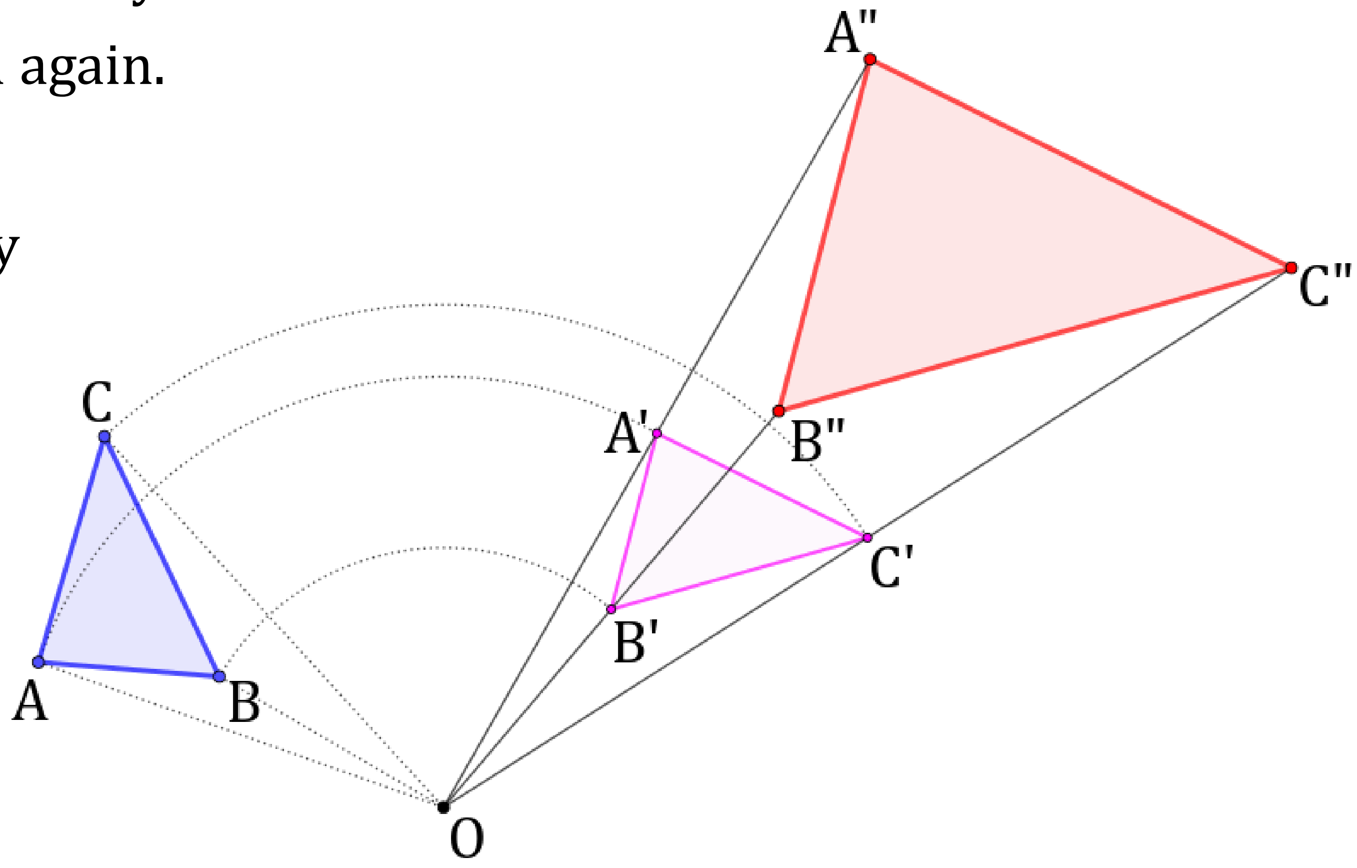
Applications of similarities 1.

We can combine a homothety and an isometry.

The result is a similarity transformation again.

For example, a rotation and a homothety with the same center:

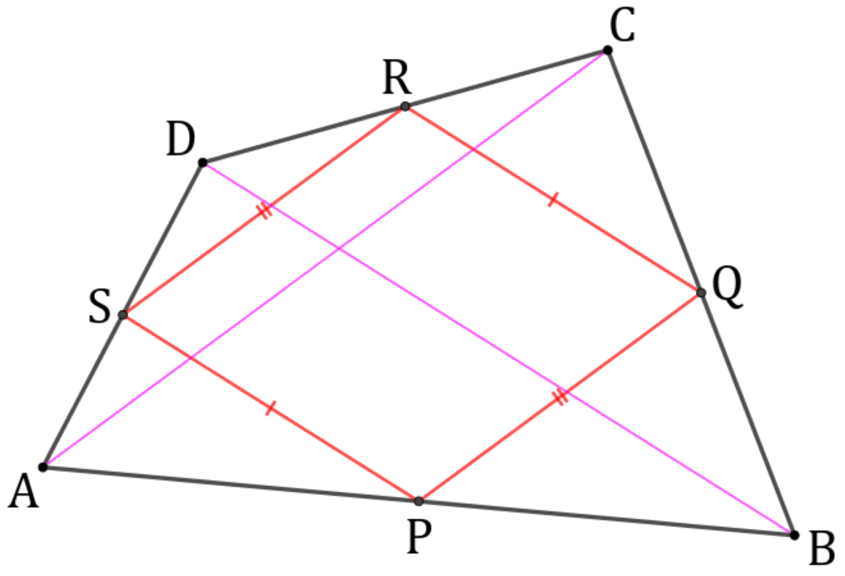
⟶ *homothety-rotation*

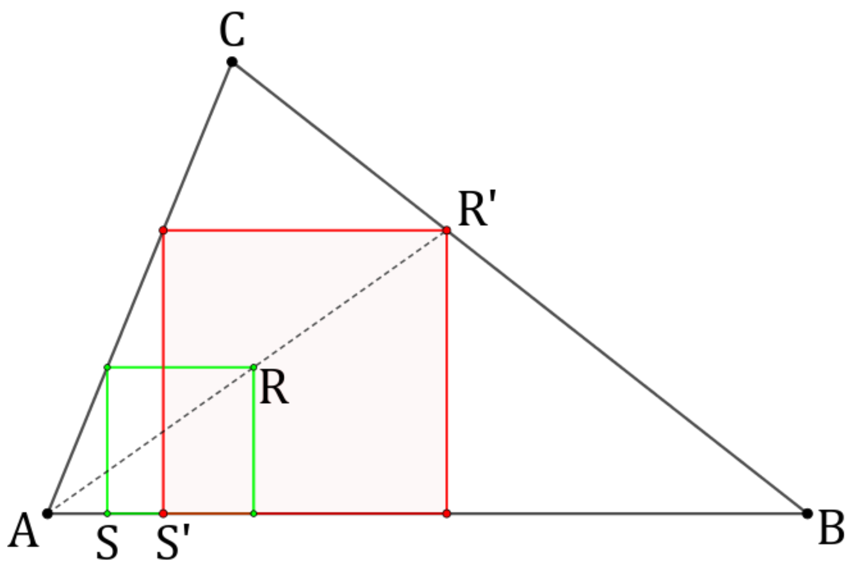


Applications of similarities 2.

Some examples

1. Show that the midpoints of sides of a quadrilateral form a parallelogram.
2. Given a triangle ABC, construct a square such that two vertices lie on BC and CA respectively, and the opposite side lies on AB.



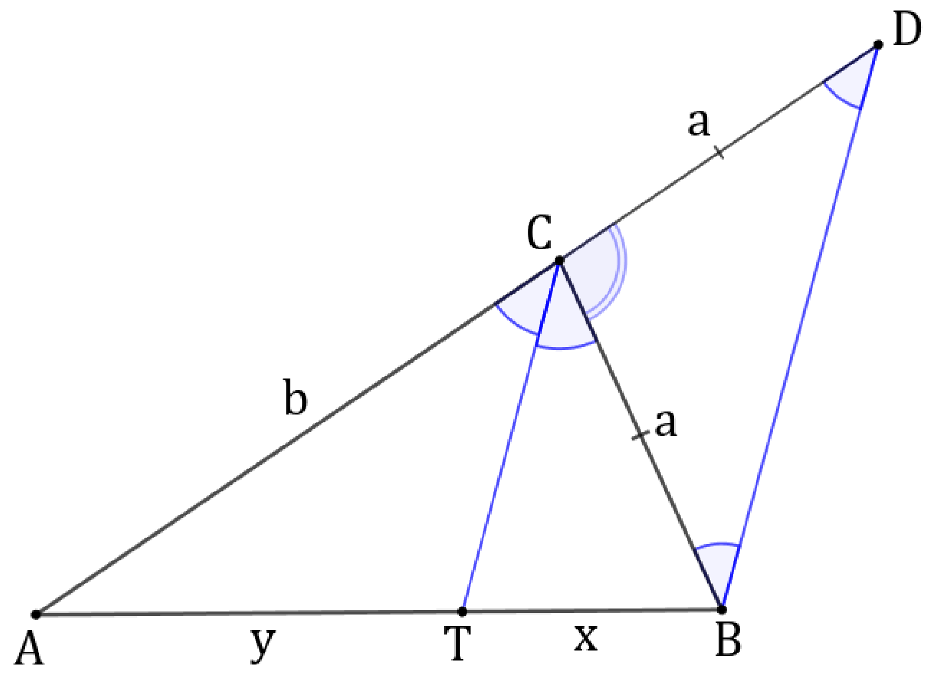


Applications of similarities 3.

3. Prove the following theorem.

Angle bisector theorem: Consider a triangle ABC. Let the angle bisector of angle C intersect side AB at a point T between A and B. Then

$$\frac{AT}{TB} = \frac{b}{a}$$



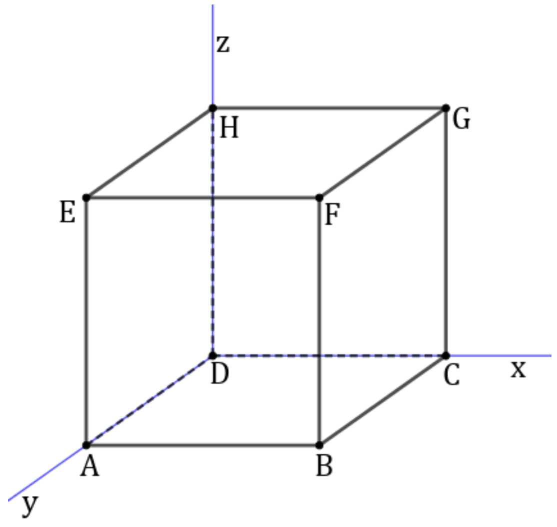
AXIAL AFFINITY

An **axial affinity** (or axial affine collineation) is a plane geometric transformation with a pointwise fixed line (axis) which preserves

- collinearity,
- parallelism,
- ratios of lengths along a line.

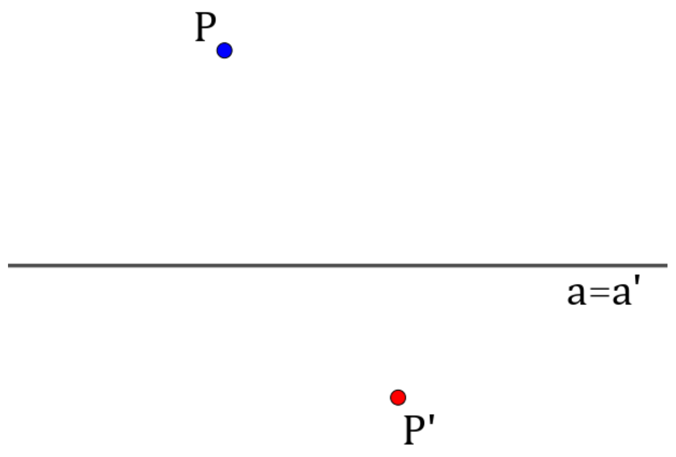
Motivation: We can find objects which are related by (axial) affinity in every parallel projection.

For example, the relation between a face of a cube and its image by a parallel projection
(e.g. face ABCD)



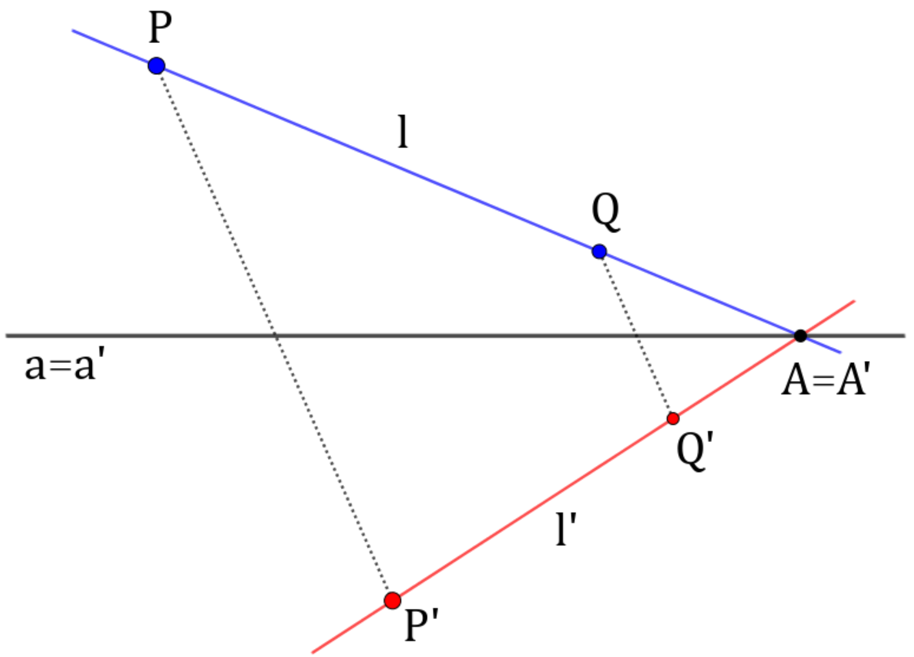
Axial affinity – construction 1.

An axial affinity is *uniquely determined* by
- the axis and
- a pair of points (P,P') where P' is the image of point P.



Constructing the image of a new point

1. Q is a new (arbitrary) point
2. drawing line PQ (line l)
3. l intersects the axis in point A
4. A is a fixed point: A=A'
5. the image of l is determined by P' and A' $\Longrightarrow$ P'A'=l'
6. copying the ratio of PQ to QA onto line l' using the intercept theorem $\Longrightarrow$
   $\Longrightarrow$ drawing a parallel line to PP' in point Q
7. the intersection point of parallel line and l' is point Q'



Axial affinity – construction 2.

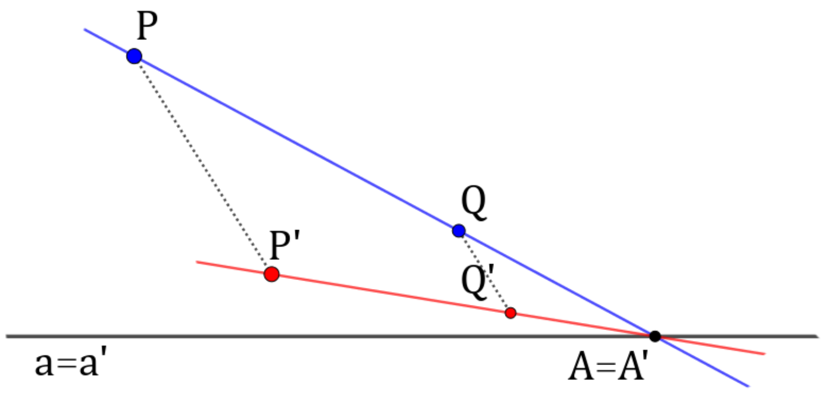
The key step of the previous construction is copying the ratio of PQ to QA. In this step, we use the definiton: an axial affinity preserves ratios of lengths along a line.

We can see that QQ’ is parallel to PP’. The direction of PP’ does not change and we use this direction to construct the image point of an arbitrary point.

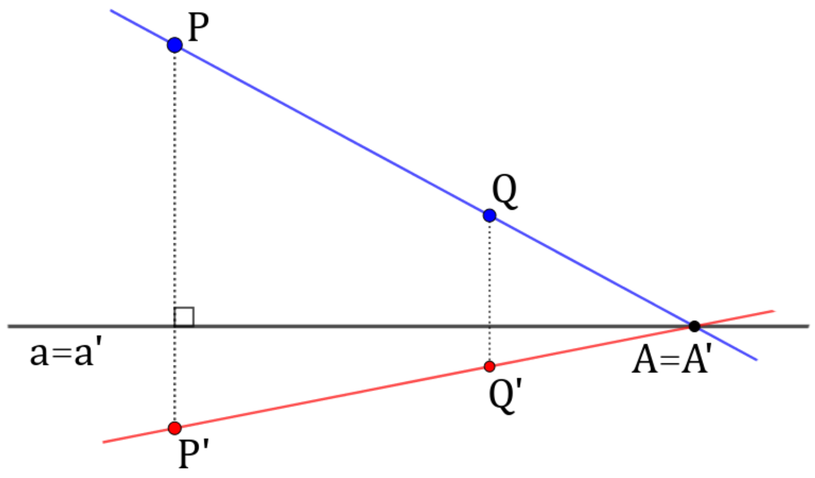
This direction is called **the direction** of the axial affinity.

The axial affinity is called *orthogonal* if the direction PP’ is perpendicular to the axis.

Another axial affinity



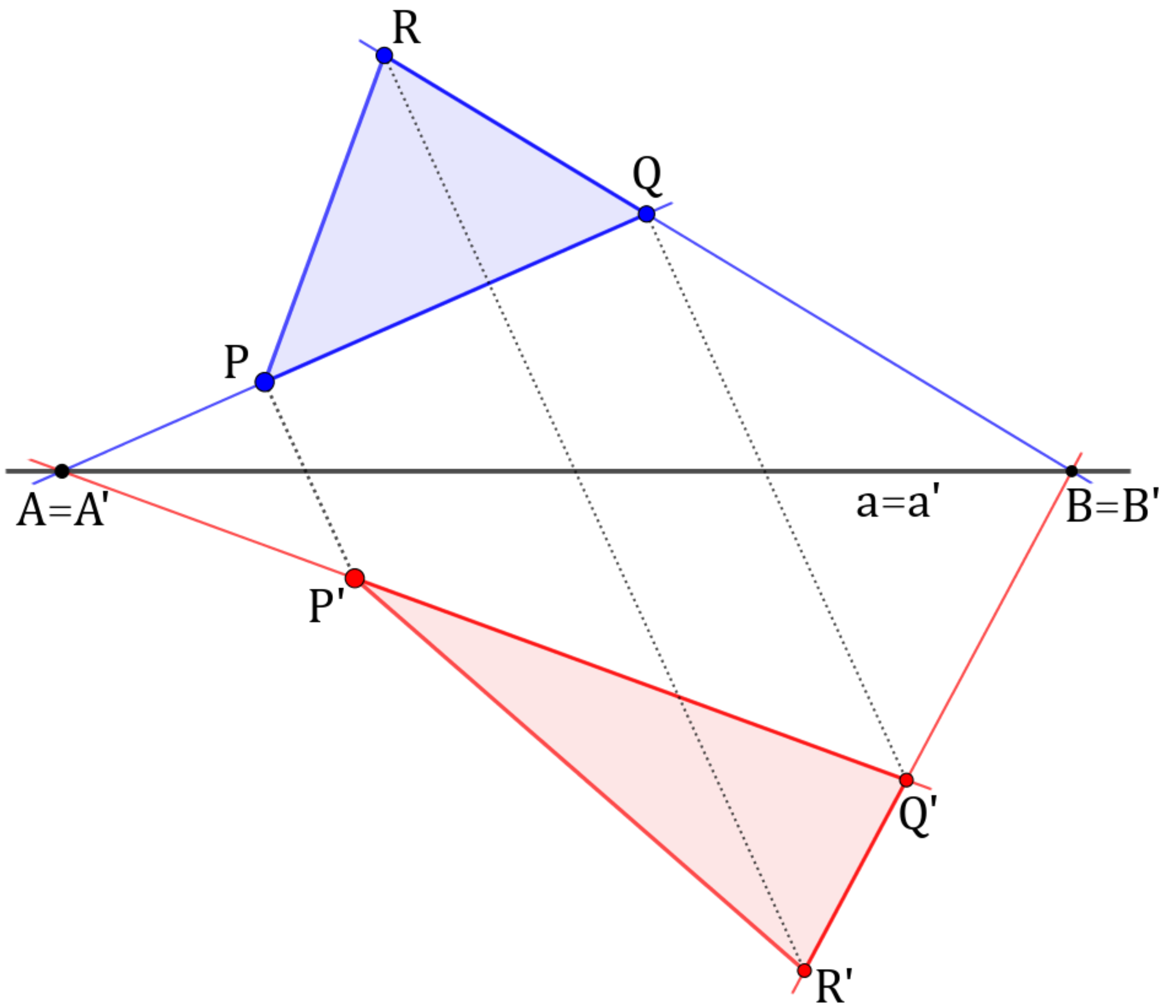
An orthogonal axial affinity



The image of a triangle

Let the axial affinity be determined by axis a and a pair of points (P,P’). Construct the image of triangle △PQR.

Hint: Use the construction of the image of a new point.



lecture

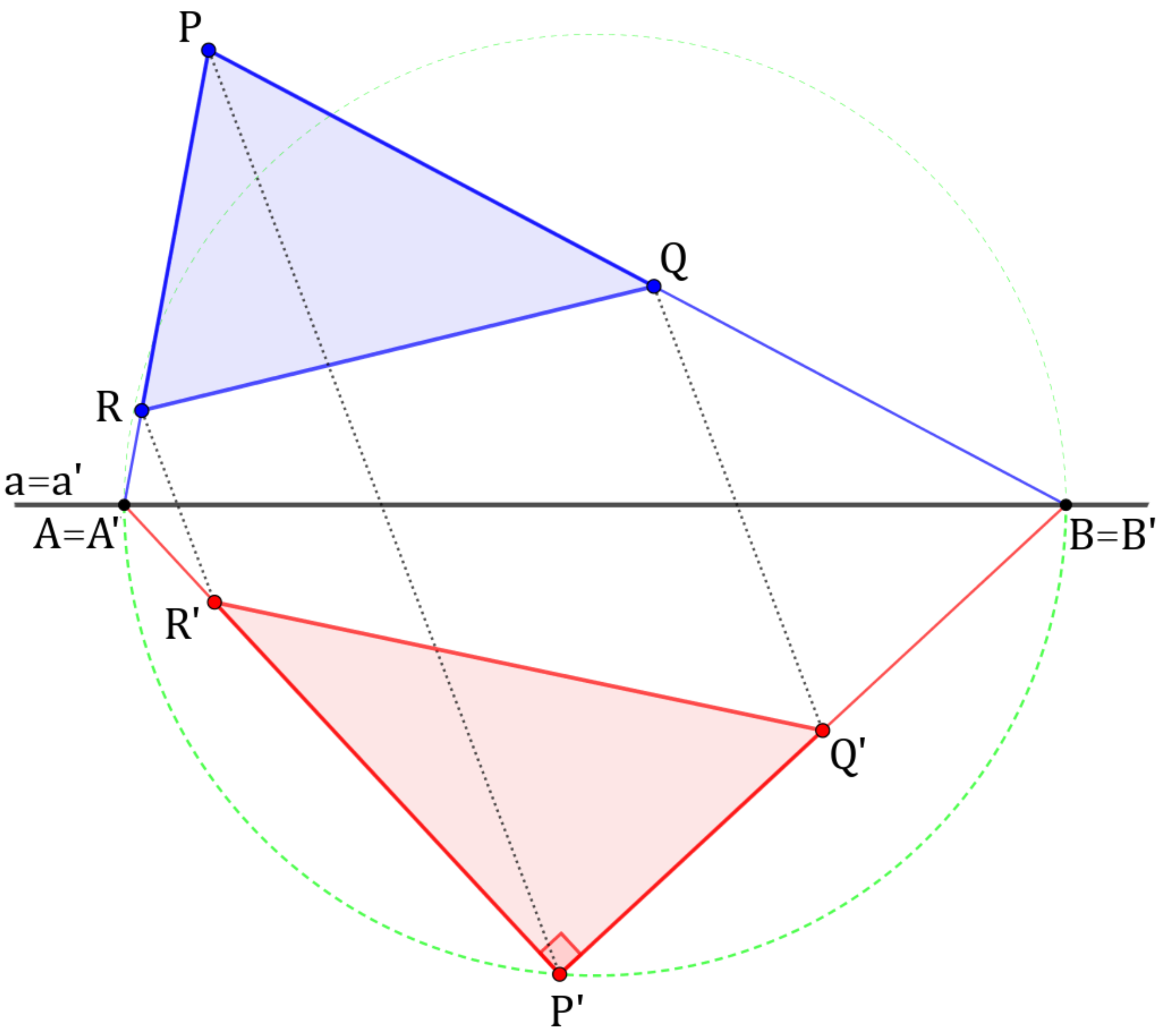
Example 1.

An arbitrary triangle △PQR and the axis of an axial affinity are given.

Determine the axial affinity so that the image of the given triangle will be a right triangle. Construct △P’Q’R’ .

Main steps

1. Goal: Let angle R’P’Q’ be a right angle.
2. The fixed points of line PR and PQ are A and B.
3. drawing the Thales’ circle over AB
4. every point of this circle can be P’
5. constructing the missing image points Q’, R’



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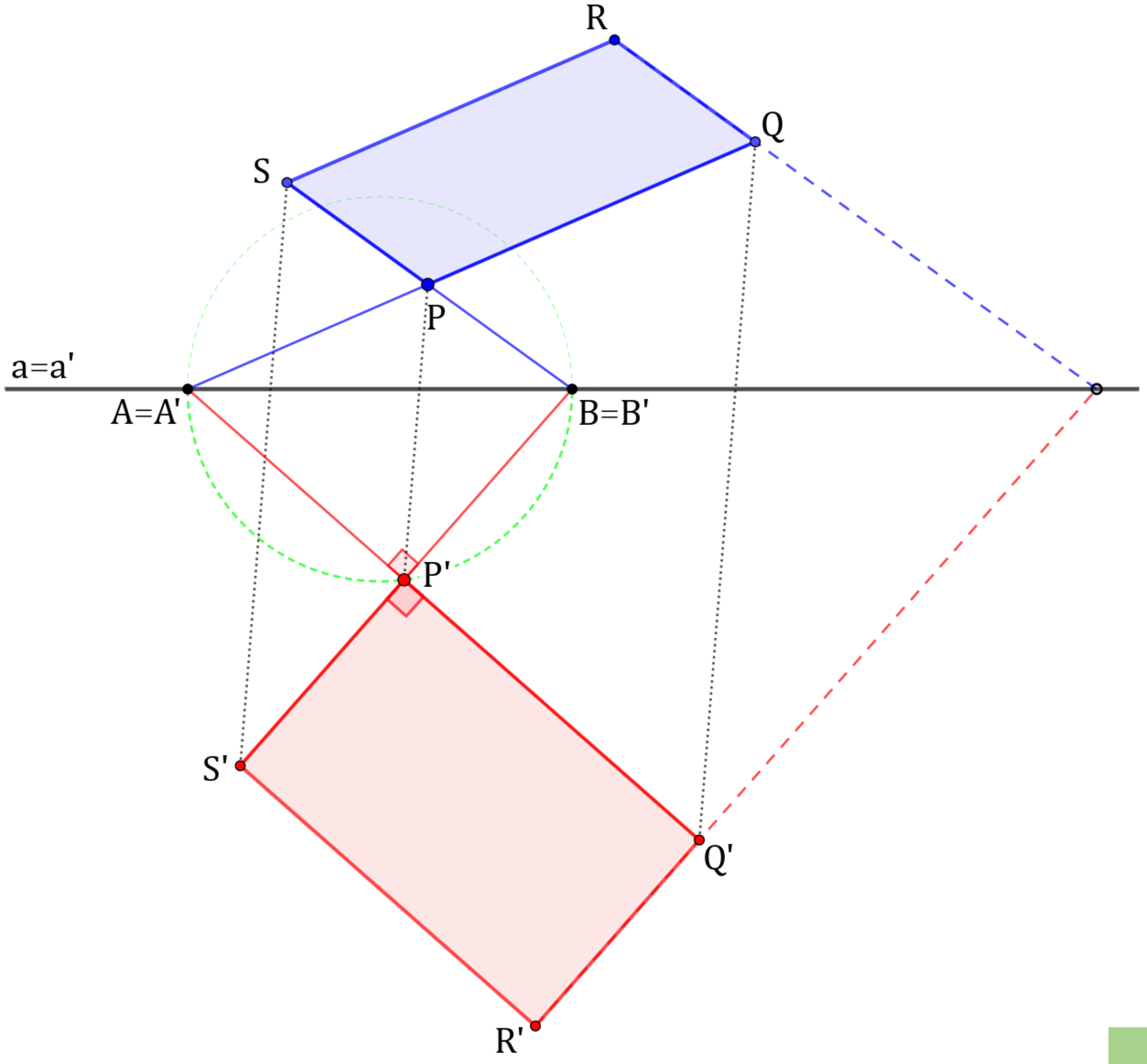
Example 2.

An arbitrary parallelogram PQRS and the axis of an axial affinity are given.

Determine the axial affinity so that the image of PQRS will be a rectangle. Construct P’Q’R’S’ .

Hint: Use Thales’ circle.



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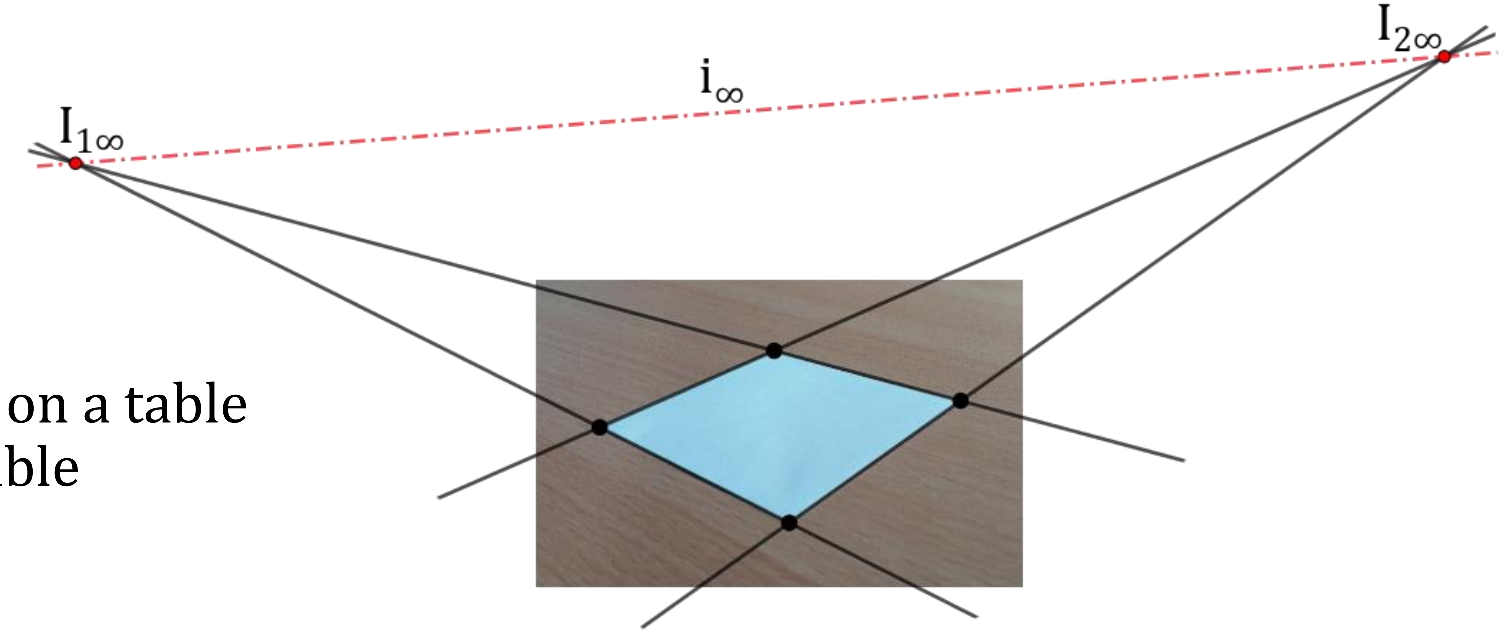
CENTRAL-AXIAL COLLINEATION

PREPARATIONS

Point at infinity (or ideal point): a new point is added to every line called its point at infinity so that parallel lines have the same point at infinity.

In a plane, every point at infinity lies on a special line called the **line at infinity** (or ideal line) of the plane.

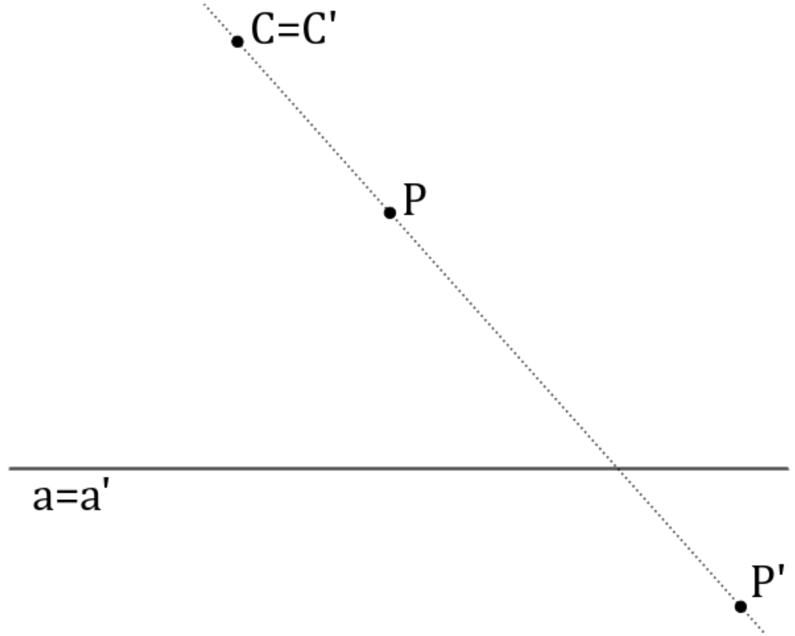
Example: a photo of a piece of paper on a table can show the line at infinity of the table



Central-axial collineation

A **central-axial collineation** (or perspective collineation) is a geometric transformation of a plane with its line at infinity which preserves collinearity, and it has a pointwise fixed line (axis) and a fixed point (center) so that every line passing through the center is an invariant line.

A central-axial collineation is *uniquely determined* by
- the axis,
- the center, and
- a pair of points (P,P’) where P’ is the image of point P so that line PP’ passes through the center.

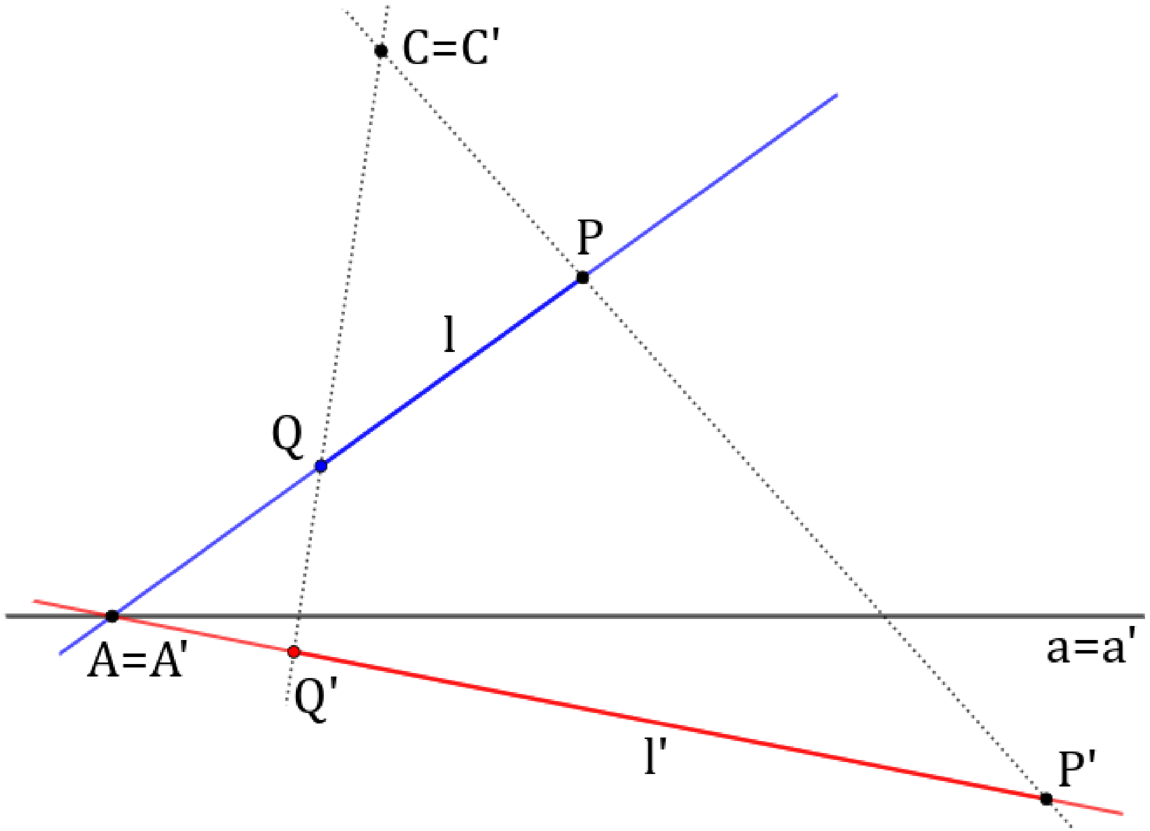


Basic constructions

A central-axial collineation is given by its axis, center and a pair of points (P,P').

Constructing the image of a new point (and the image of a line)

1. Q is a new (arbitrary) point
2. drawing line PQ (line l)
3. l intersects the axis in point A
4. A is a fixed point: A=A'
5. the image of l is determined by P' and A' $\Longrightarrow$
   $\Longrightarrow$ P'A'=l'
6. drawing line CQ (an invariant line)
7. the intersection point of lines CQ and l' is point Q'



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Vanishing line and neutral line

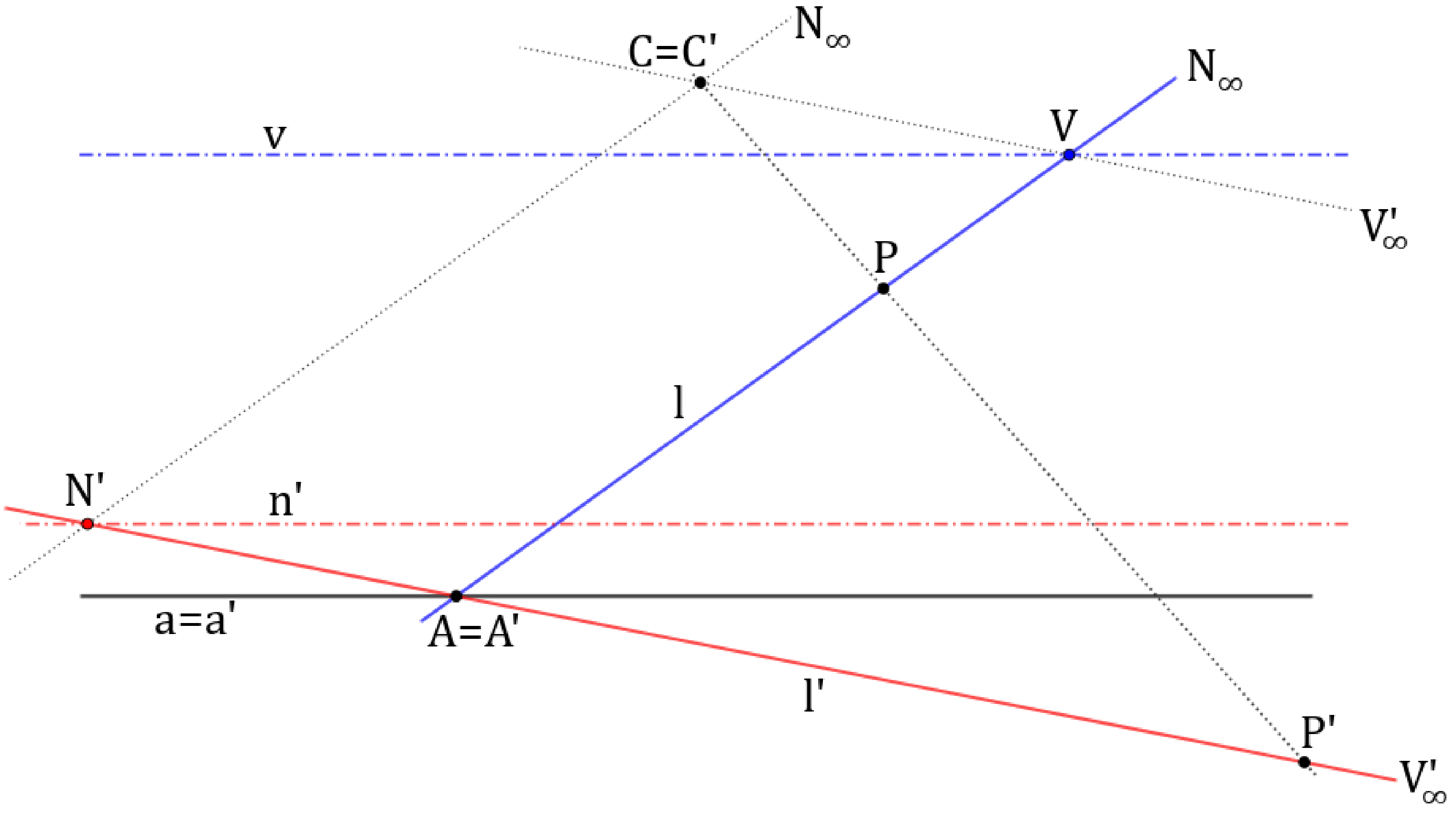
A central-axial collineation has two special lines which are parallel to the axis: the vanishing line and the neutral line.

The vanishing line (v):
its image is the line at infinity.

Each line l has one special point V (called its *vanishing point*) so that line CV is parallel to l'.
$\Longrightarrow$ Image V'_∞ is the point at infinity of l'.

The neutral line (n'):
it is the image of the line at infinity.

Each line l' has one special point N' so that line CN' is parallel to l.
$\Longrightarrow$ N' is the image of the point at infinity of line l (denoted by N_∞).



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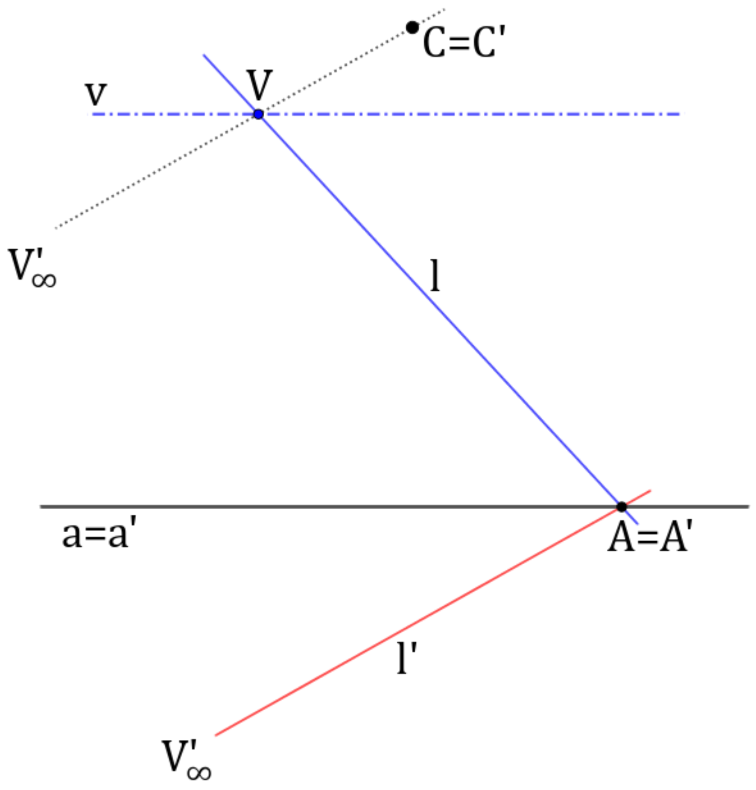
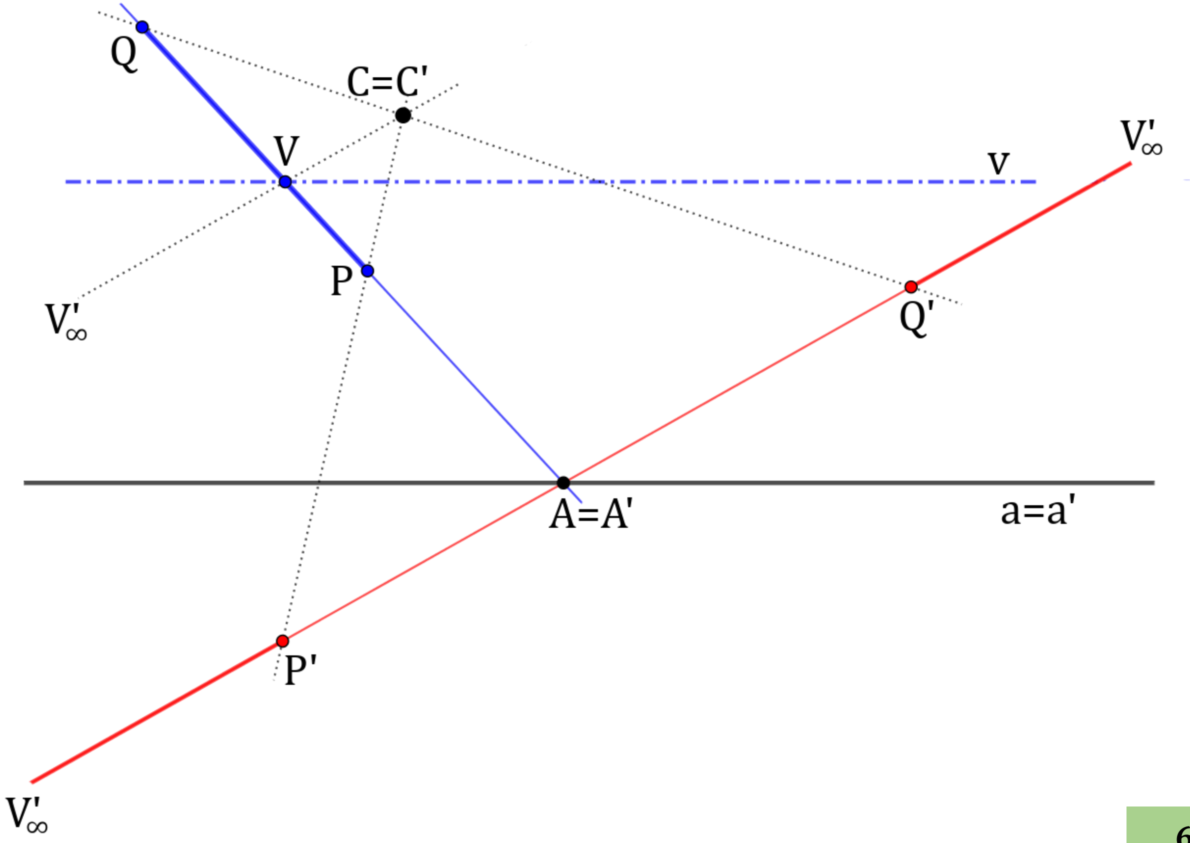
Properties of the vanishing line

A central-axial collineation is *uniquely determined* by its axis, center, and vanishing line.

Construction of the image of line l
(using its vanishing point)



Construction of the image of segment PQ
(if PQ contains the vanishing point)

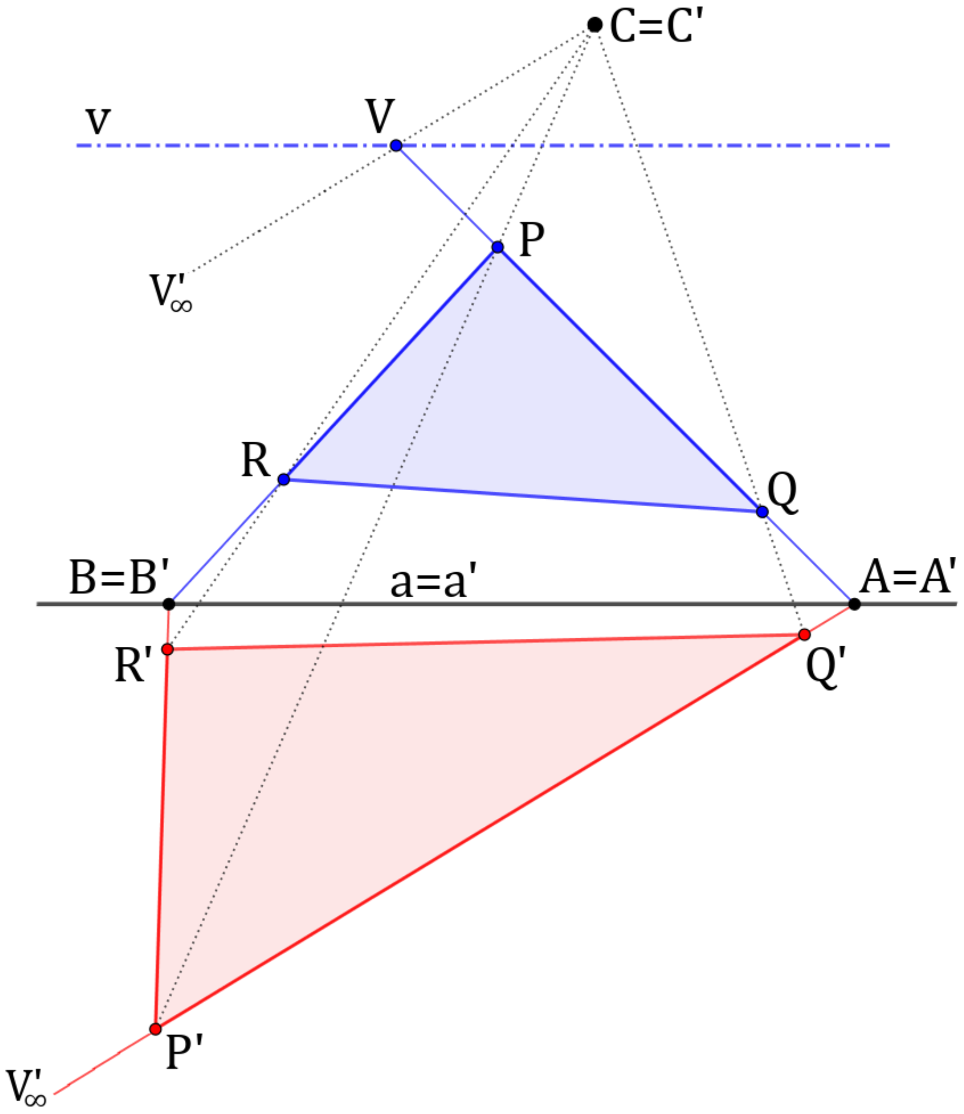


The image of a triangle 1.

The axis, center, and vanishing line
of a central-axial collineation are given.

Construct the image of triangle △PQR.

Hint: Use the vanishing point of a side.



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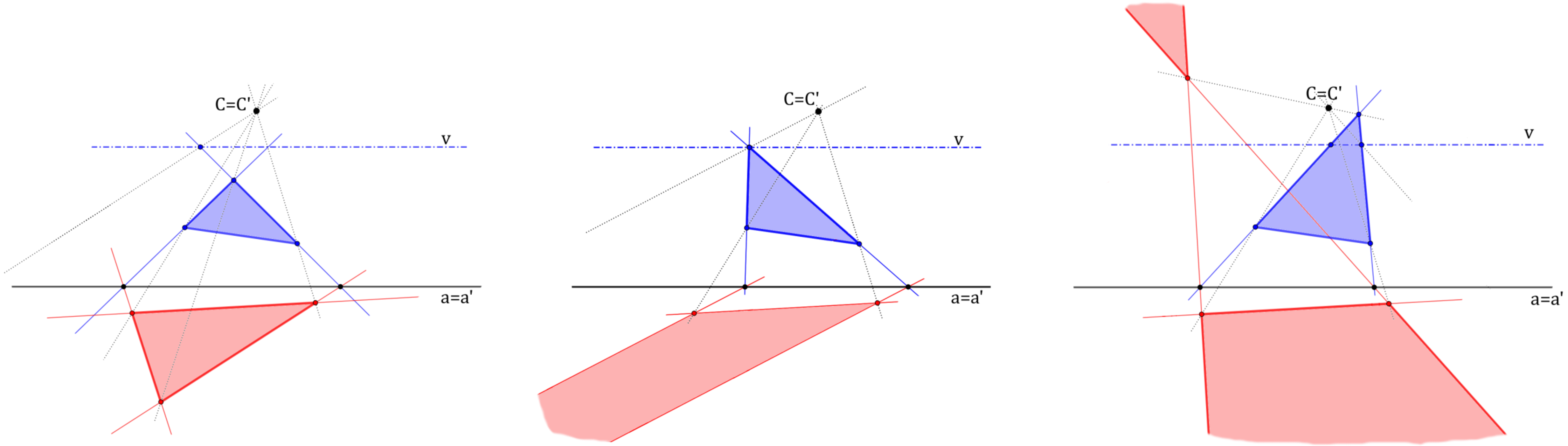
The image of a triangle 2.

Remark: The image of a triangle can be formed by segments and rays as well.



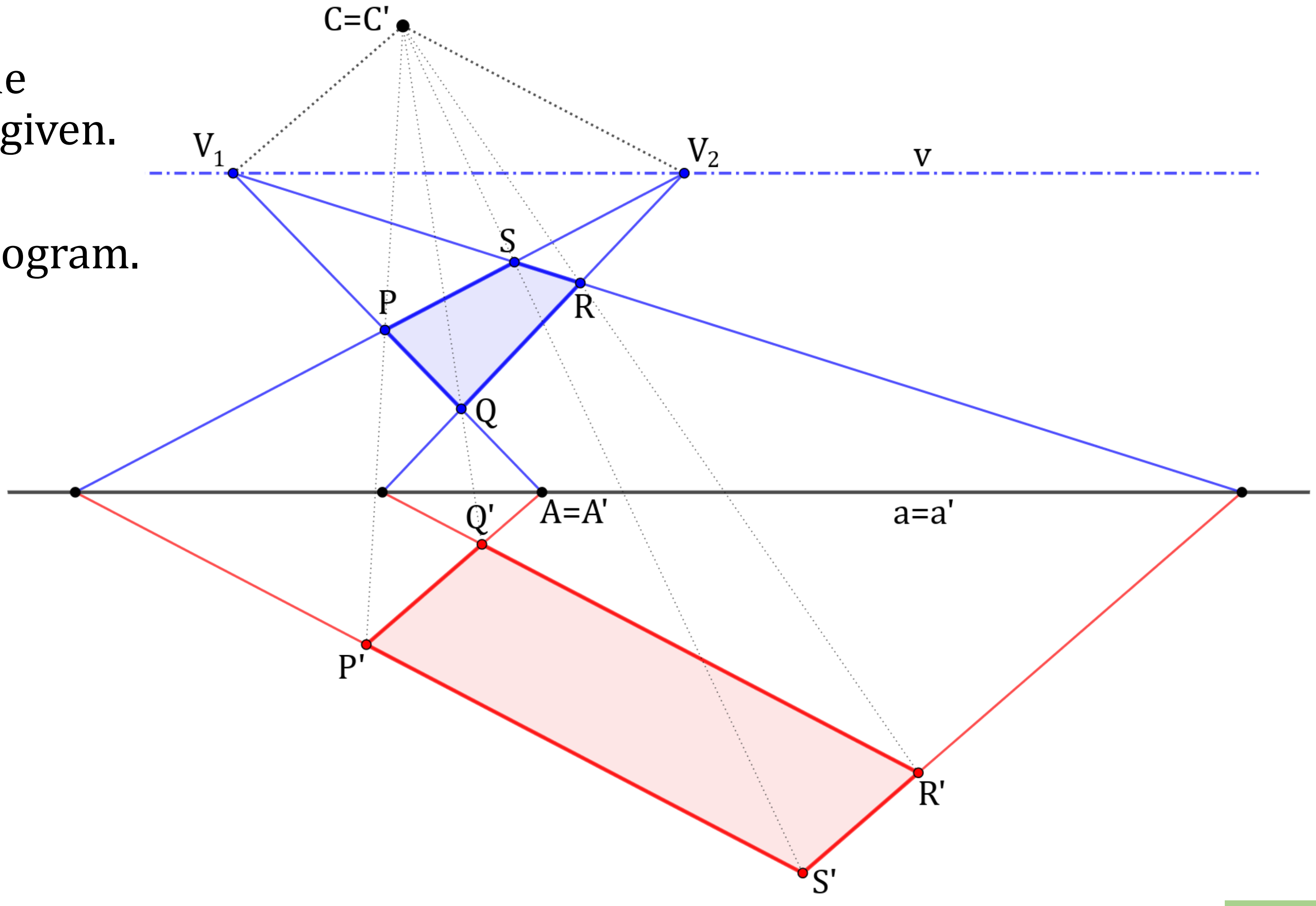
The shape of the image depends on the number of intersecting points of the triangle and the vanishing line.

The image of a special quadrilateral

The axis, center, and vanishing line
of a central-axial collineation are given.

Construct a quadrilateral PQRS
so that its image will be a parallelogram.

Construct the image of PQRS.



Hint: Use the vanishing line.

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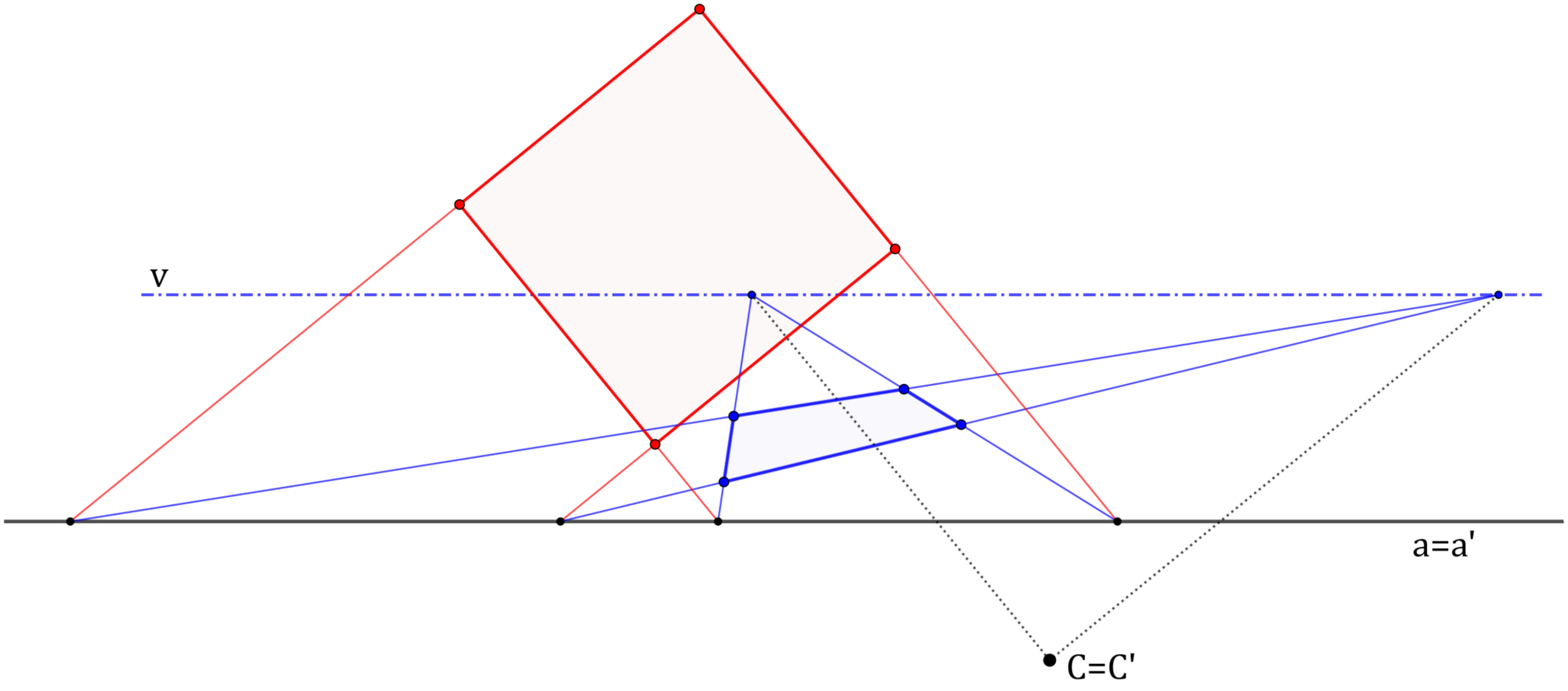
Application 1.

In this case, the image of a quadrilateral is a square

⟶ constructions in a perspective system (see Descriptive Geometry) / freehand drawing



Application 2.

The construction of the intersection of a pyramid and a plane

⟶ see Descriptive Geometry (next academic year)