GEOMETRICAL CONSTRUCTIONS 2

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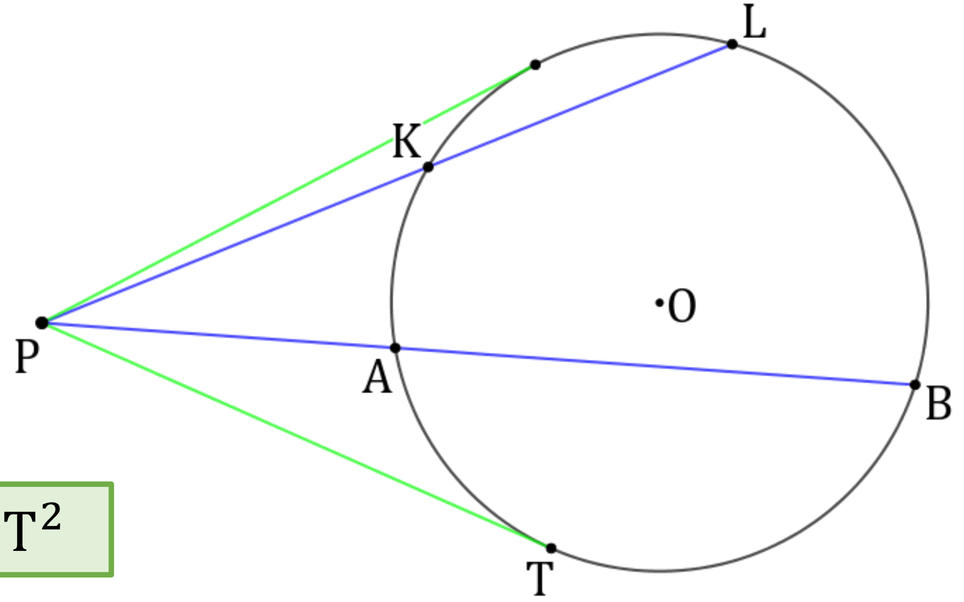
Power of a point

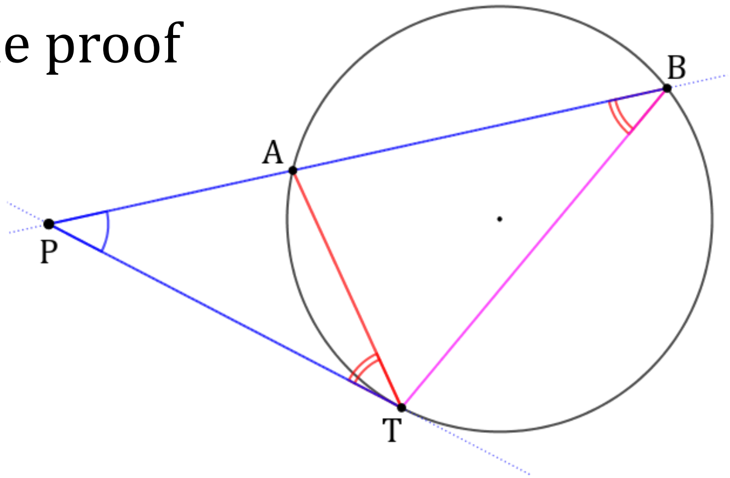
Power of a point (or circle power)
The power of a point P with respect to a circle is defined by the product $PA \cdot PB$ where AB is a secant passing through P.

Remark: The point P can be an inner point of the circle.



Theorem: Using the notation of the figure, $PA \cdot PB = PK \cdot PL = PT^2$

Sketch of the proof



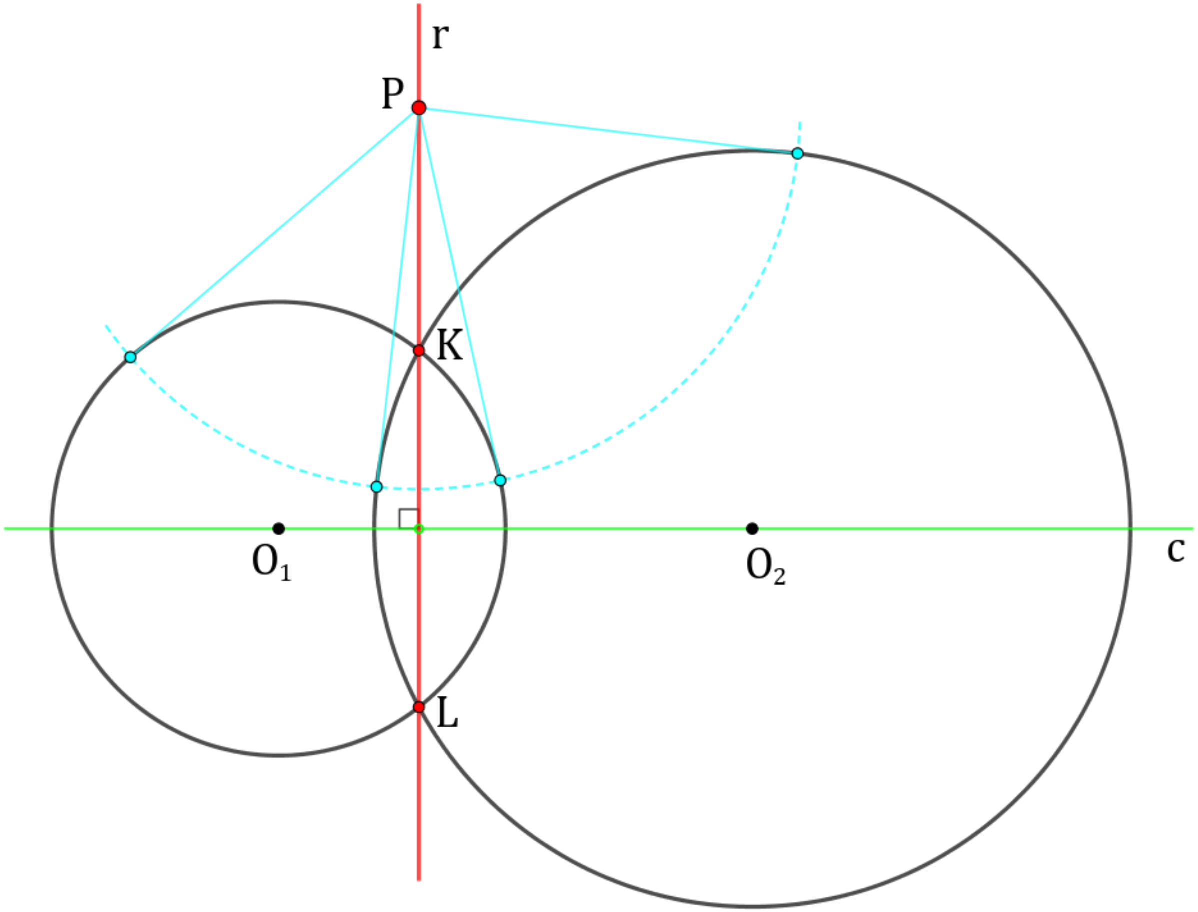
RADICAL LINE, RADICAL CENTER – 1

Radical line (of two circles)
is the locus of points of equal (circle) power with respect to two nonconcentric circles.

This line is perpendicular to the line of centers.

If two circles have common points K, L then their radical line is line KL.

Alternative definition:
radical line is the locus of points at which tangents drawn to both circles have the same length. (See point P.)



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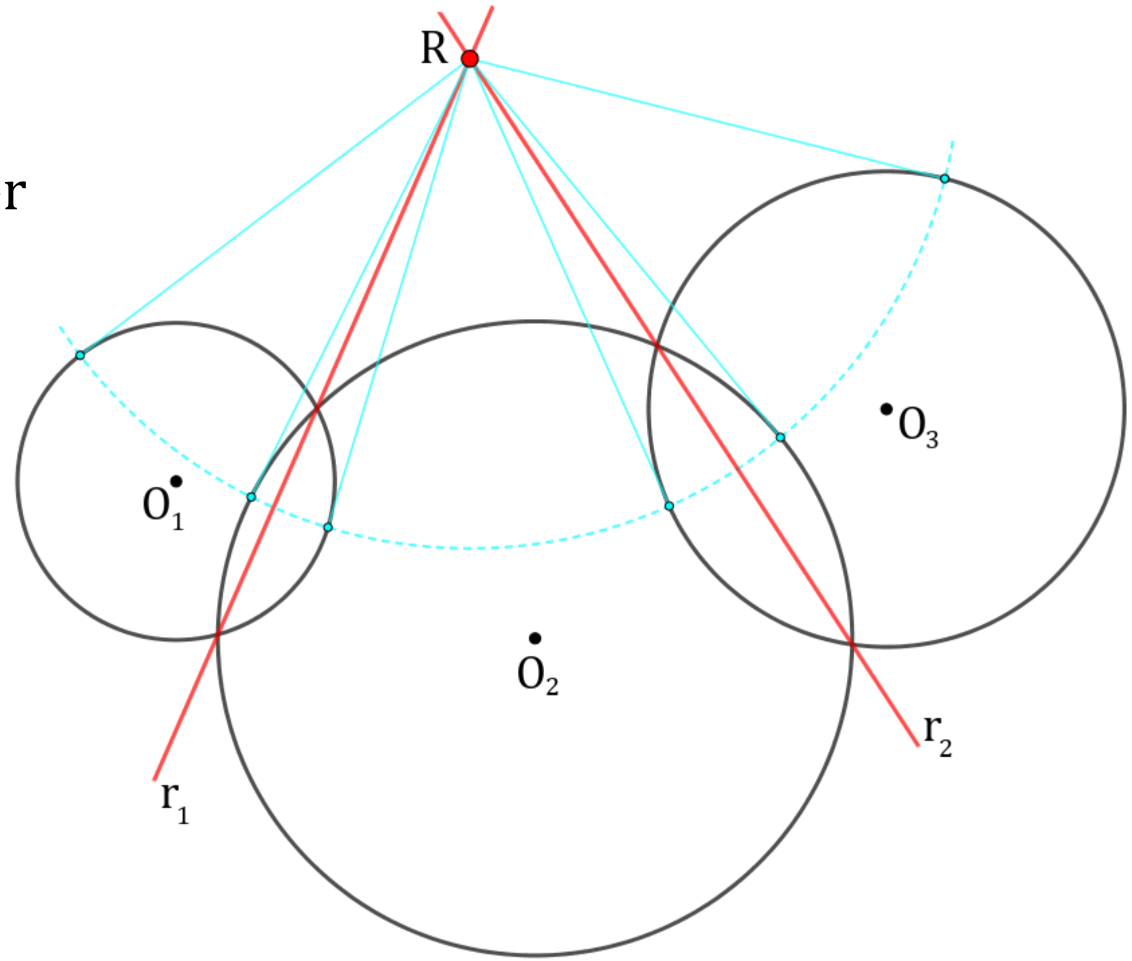
Radical line, radical center – 2

Radical center (of three circles):
The radical lines of three circles intersect each other in a special pont called the radical center (or power center).



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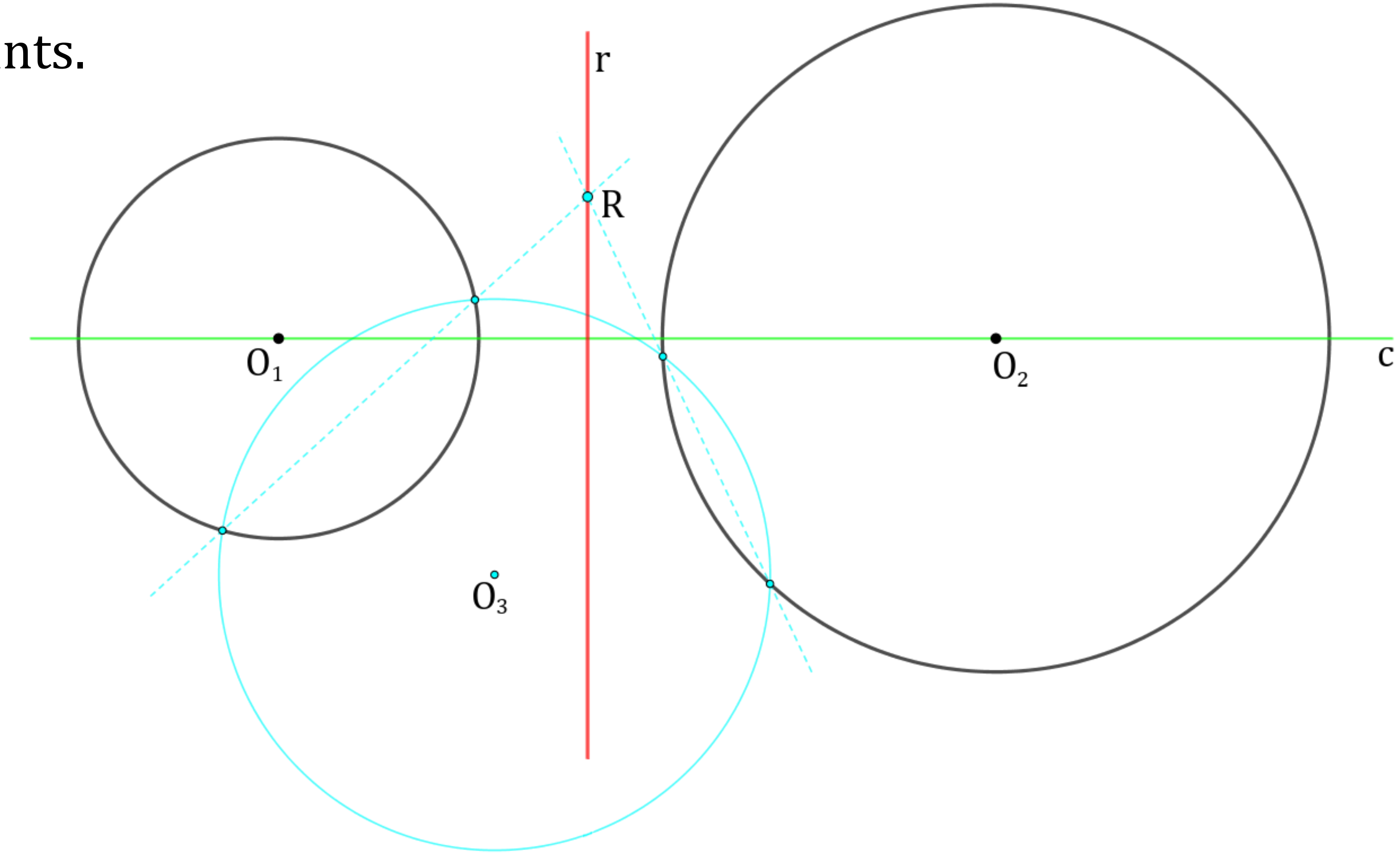
Radical line, radical center – 3

The radical line exists even if
the two circles have no common points.

We can construct this line using
a third circle which intersects
both of circles.

(→ using the radical center of three circles)



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APOLLONIUS’ PROBLEM

THE PROBLEM: Given three objects, each of which may be a point, a line, or a circle. Construct circles which are tangent to the given objects.

There are 10 different cases:

1. 3 points (max. 1 solution)
2. 2 points, 1 line (max. 2 solutions)
3. 2 points, 1 circle (max. 2 solutions)
4. 1 point, 2 lines (max. 2 solutions)
5. 1 point, 1 line, 1 circle (max. 2 solutions)
6. 1 point, 2 circles (max. 4 solutions)
7. 3 lines (max. 4 solutions)
8. 2 lines, 1 circle (max. 4 solutions)
9. 1 lines, 2 circle (max. 4 solutions)
10. 3 circles (max. 8 solutions)

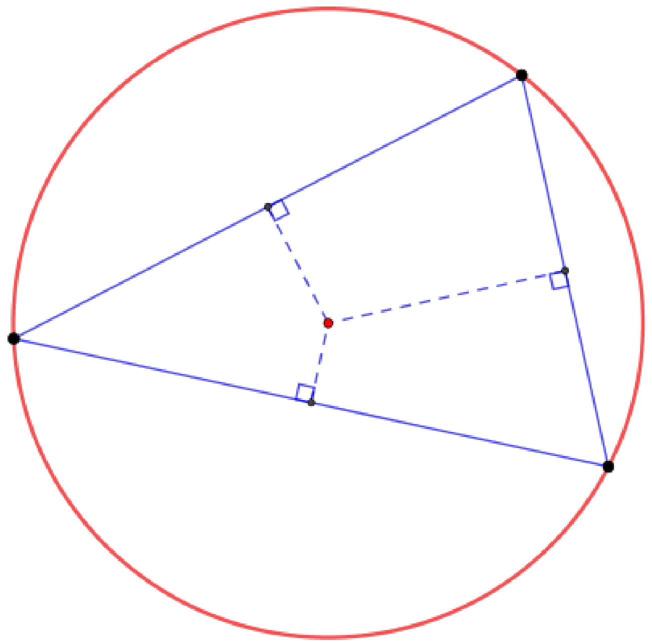
Important remark:
The number of solutions, and ideas of constructions are depend on relations of given elements.

E.g. If two points and a line are given so that given points lie on different sides of the line, we cannot construct tangent circles.

Apollonius’ problem – trivial cases

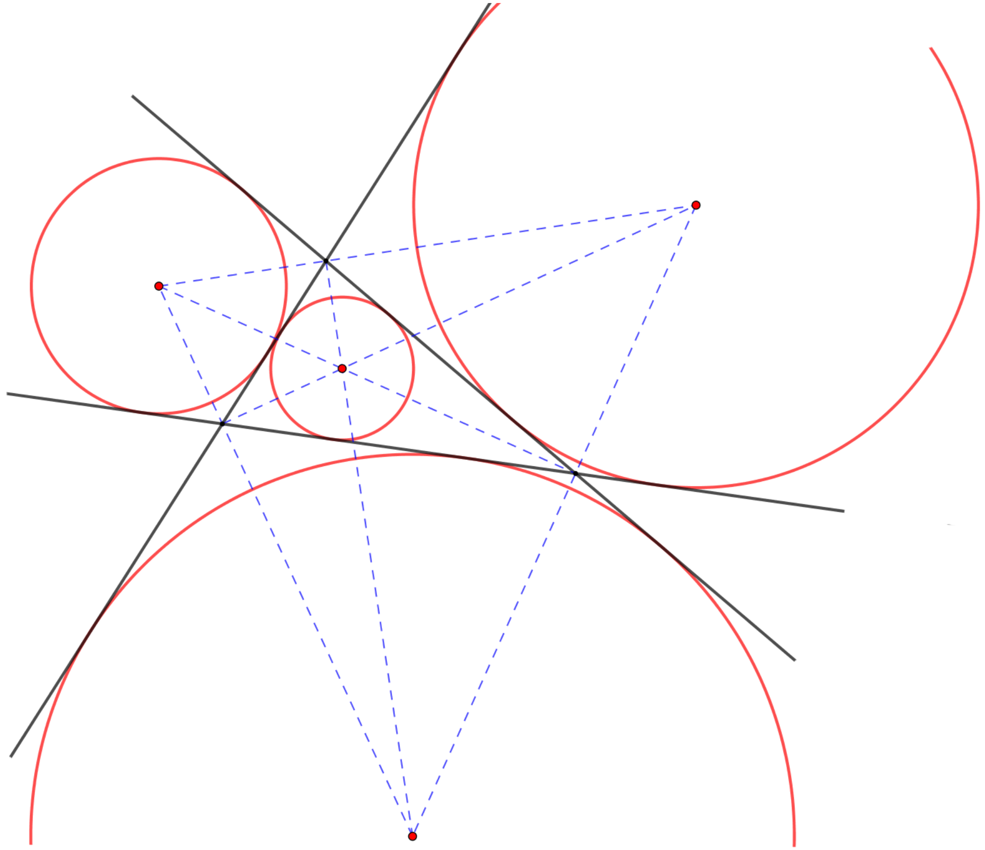
3 points

= circumscribed circle of a triangle
(see Geometrical Constructions 1)



3 lines

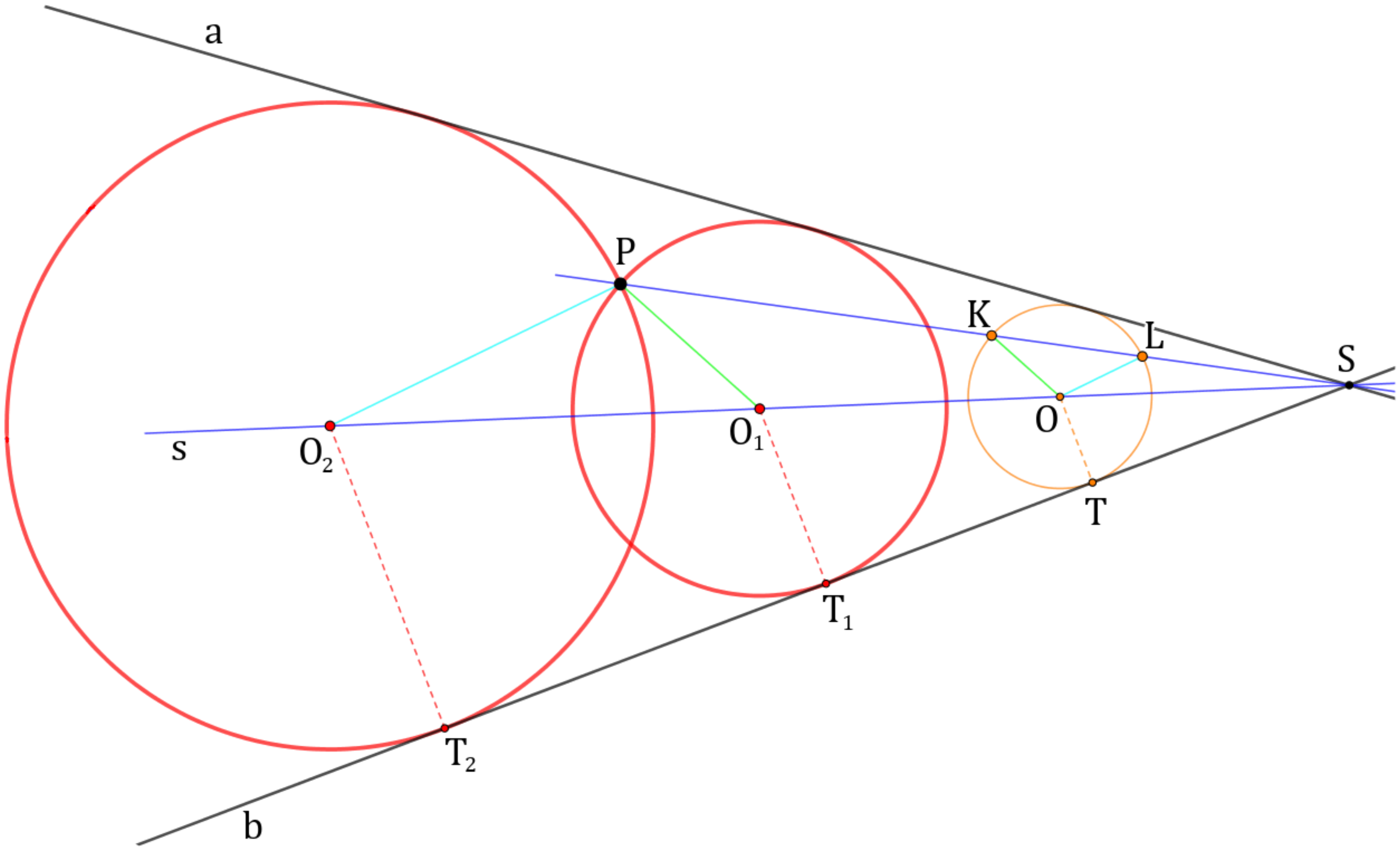
= incircle and excircles of a triangle
(see Geometrical Constructions 1)



Apollonius' problem – 1 point, 2 lines

Sketch of the construction:

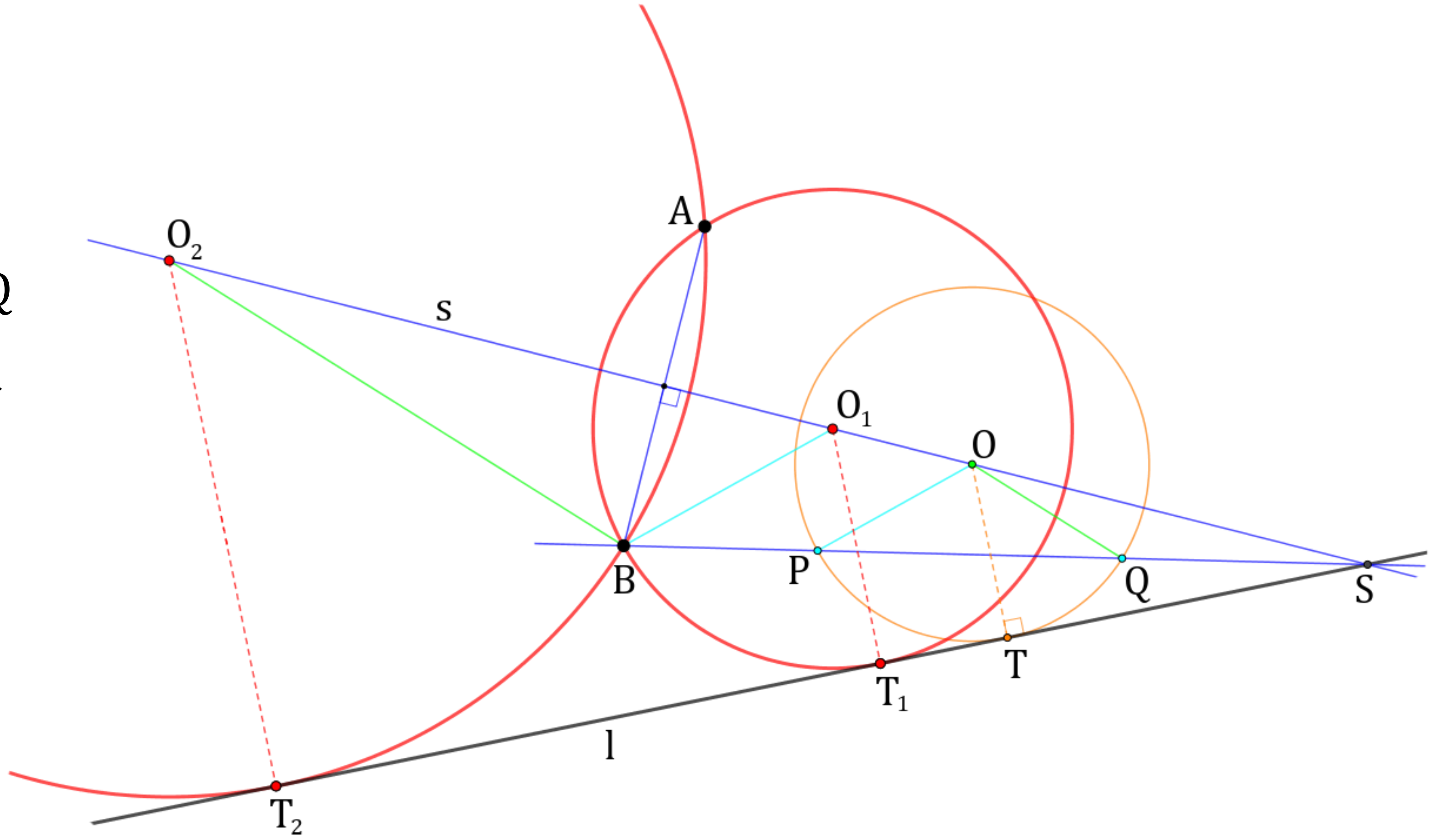
1. angle bisector of a and b is s
   (It contains the centers of solutions.)
2. center O lies on s, OT is the radius
   → drawing a circle
3. line PS intersects this circle: K, L
4. drawing parallel lines to OK and OL passing through P
   (using homothety)
5. these lines intersect line s
   → centers O_1 and O_2
6. two solutions:
   circle 1 – center O_1, radius O_1T_1
   circle 2 – center O_2, radius O_2T_2



Apollonius' problem – 2 points, 1 line

Sketch of the construction:

1. bisector of AB is s
   (It contains the centers of solutions.)
2. center O lies on s, OT is the radius
   → drawing a circle
3. e.g. line BS intersects this circle: P, Q
4. drawing parallel lines to OP and OQ passing through B
   (using homothety)
5. these lines intersect line s
   → centers O_1 and O_2
6. two solutions:
   circle 1 – center O_1, radius O_1T_1
   circle 2 – center O_2, radius O_2T_2

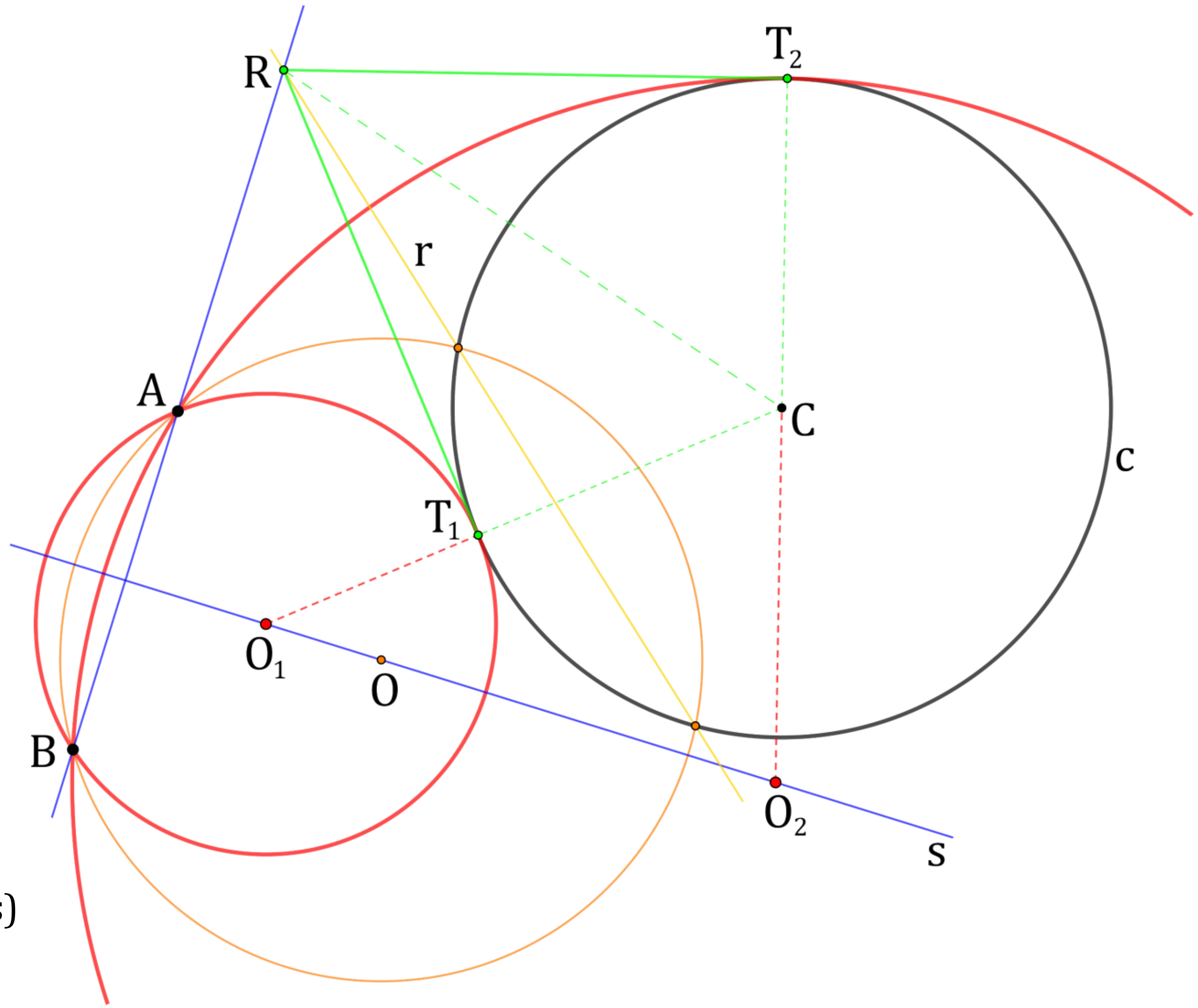


practical (Alternative solution: We can use radical lines as well.)

Apollonius' problem – 2 points, 1 circle

Sketch of the construction:

1. bisector of AB is s
2. drawing an arbitrary circle with center O so that it intersects the given circle (O lies on s)
3. the radical line of c and „orange" circle is r
4. line AB is the radical line of two circles which we would like to construct
5. the intersection point of AB and r is R (radical point)
6. constructing tangent lines to the given circle from point R $\rightarrow$ points T_1 and T_2 (Hidden idea: We can move secant r until r becames a tangent line. So the arbitrary, orange circle becames a tangent circle.)
7. two solutions:
   circle 1 – center O_1, radius O_1T_1
   circle 2 – center O_2, radius O_2T_2
   (constructing a circle with 3 given points, or tangent lines)



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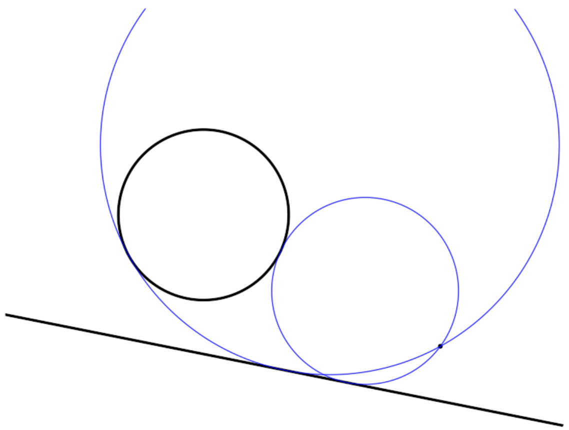
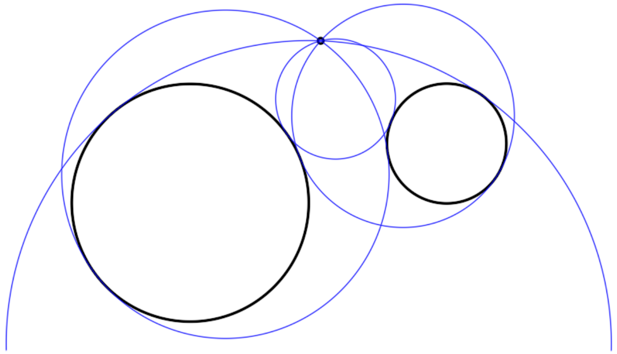
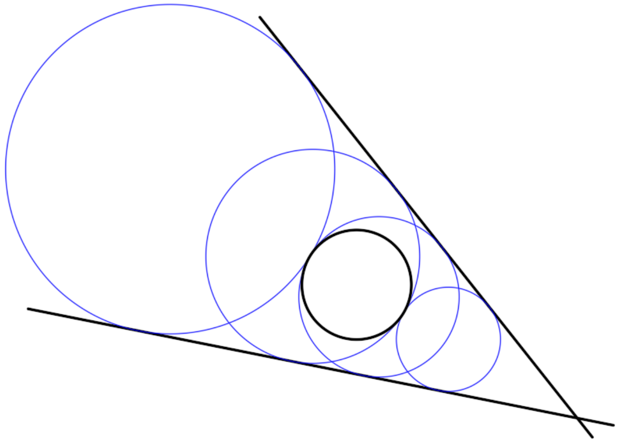
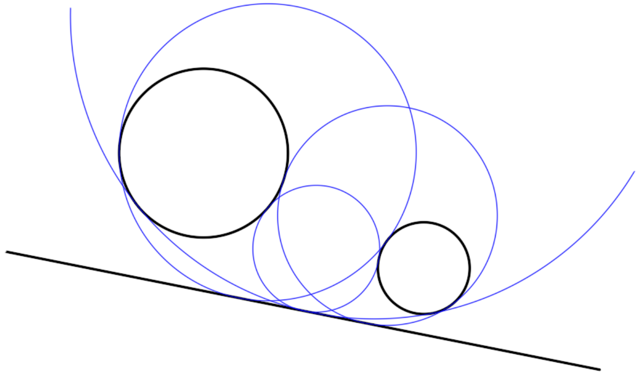
Apollonius’ problem – other cases

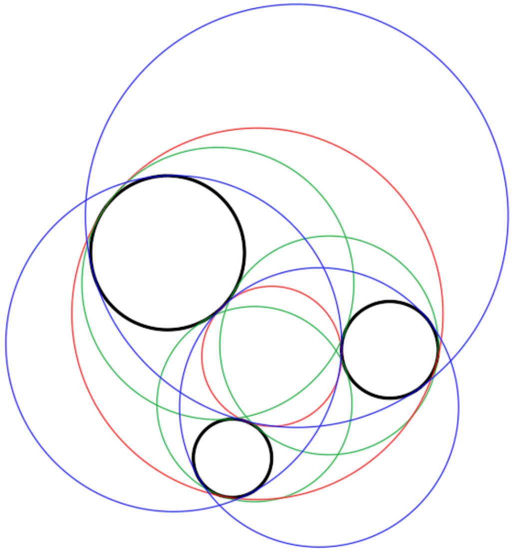
In remaining cases, finding solutions is not quite so easy.









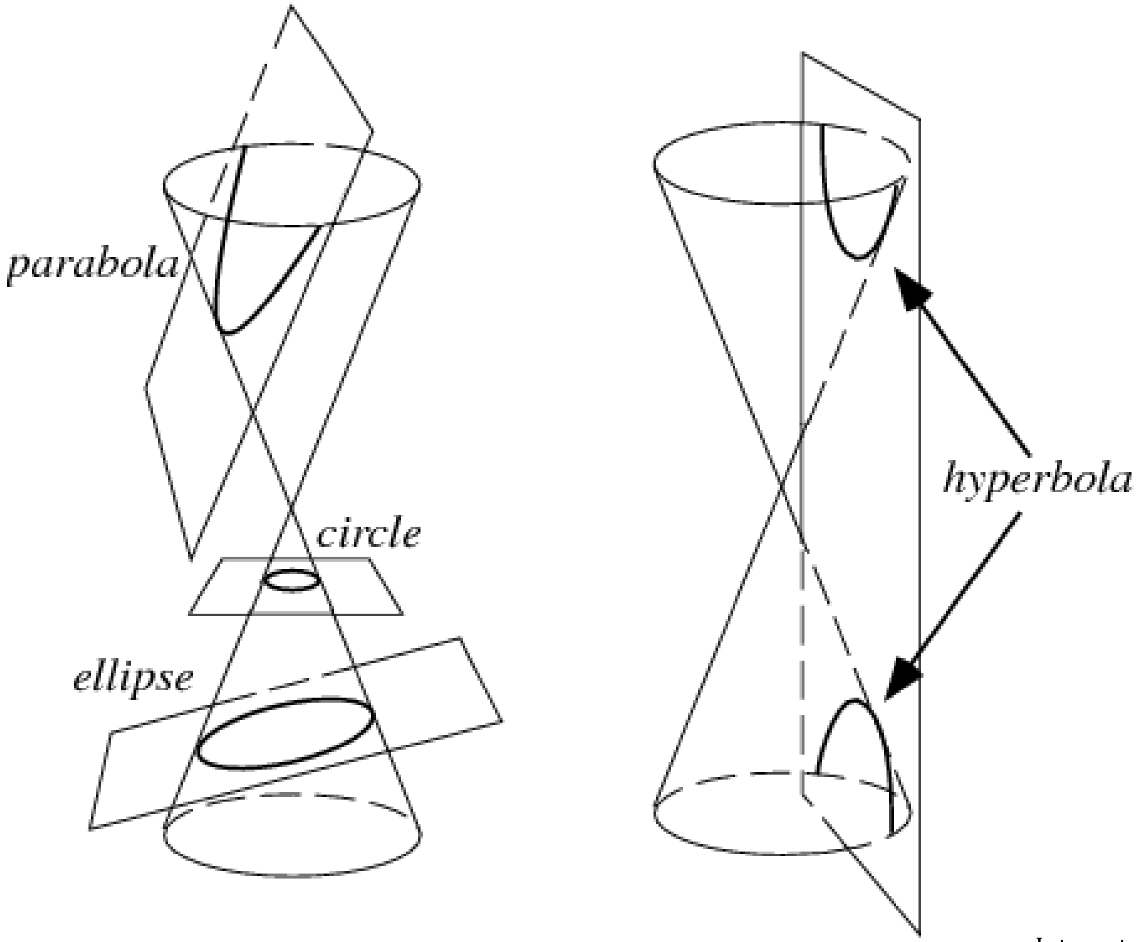


CONIC SECTIONS

Conic section: a curve generated by the intersection of a plane and a cone

A conic section can be

- *a circle or an ellipse*
- *a parabola*
- *a hyperbola*
- (one or two lines)
- (one point)



source: Internet

Another definition:
a conic section is the set of points whose coordinates satisfy a quadratic equation in two variables.

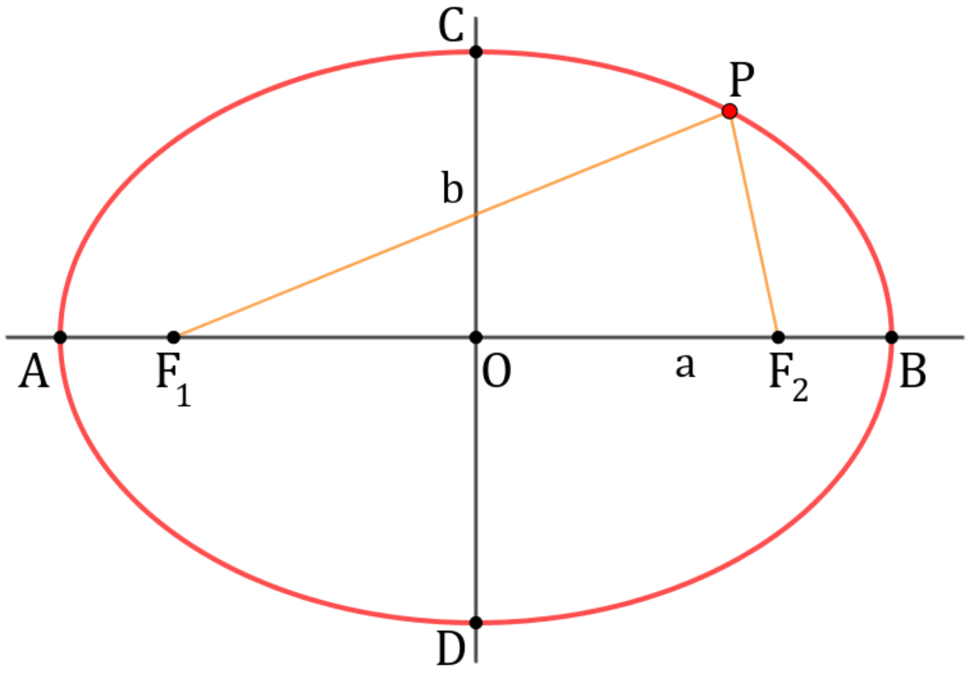
Ellipse

Ellipse: the locus of points in a plane with respect to two given points so that the sum of the distances to the two points is constant for every point on the curve.

$\mathbf{F_1P+F_2P = constant} = 2a > F_1F_2$

F_1, F_2 – *foci*
O – *center*
A, B – *vertices*
C, D – *co-vertices*

AB = 2a – *major axis* (OA, OB – semi-major axes)
CD = 2b – *minor axis* (OC, OD – semi-minor axes)



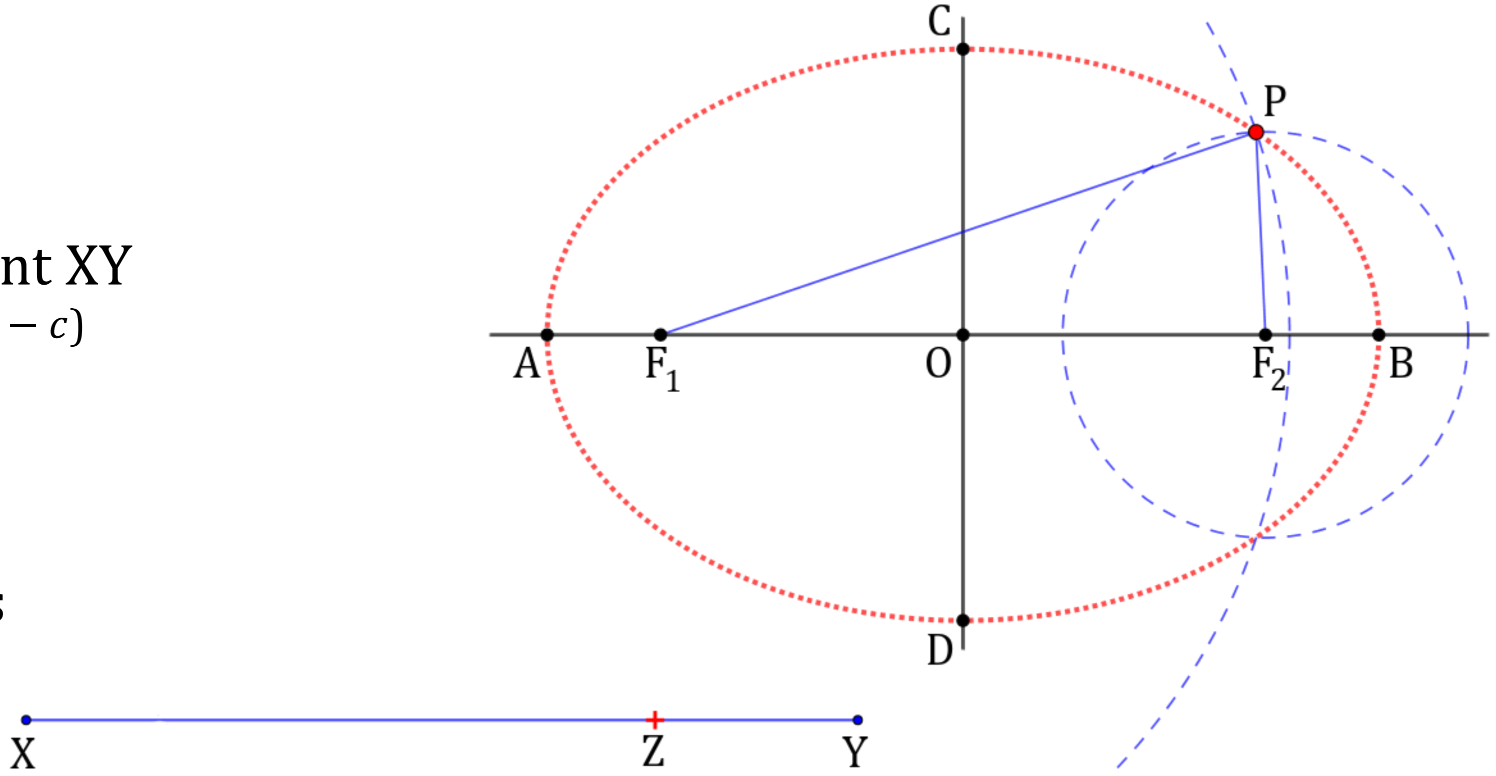
The axes of an ellipse (AB and CD) determine the ellipse as well.

Remark: If $F_1=F_2$, we obtain a circle.

Ellipse – construction of its points

The length of the major axis (2a) and two foci are given.

Construct an arbitrary point of the ellipse.



Using the definition,

1. XY = 2a ($2a > F_1F_2$)
2. Z is an arbitrary point on segment XY
   ($XZ \geq \frac{XY - F_1F_2}{2} = a - c$, or $YZ \geq \frac{XY - F_1F_2}{2} = a - c$)
3. drawing two circles:
   center F_1, radius XZ
   center F_2, radius YZ
4. the intersection points of circles
   are points of the ellipse
   ($F_1P + F_2P = 2a$)

lecture

Ellipse – further terminology

OF_1, OF_2 – *linear eccentricity* (c)

$b^2 + c^2 = a^2$

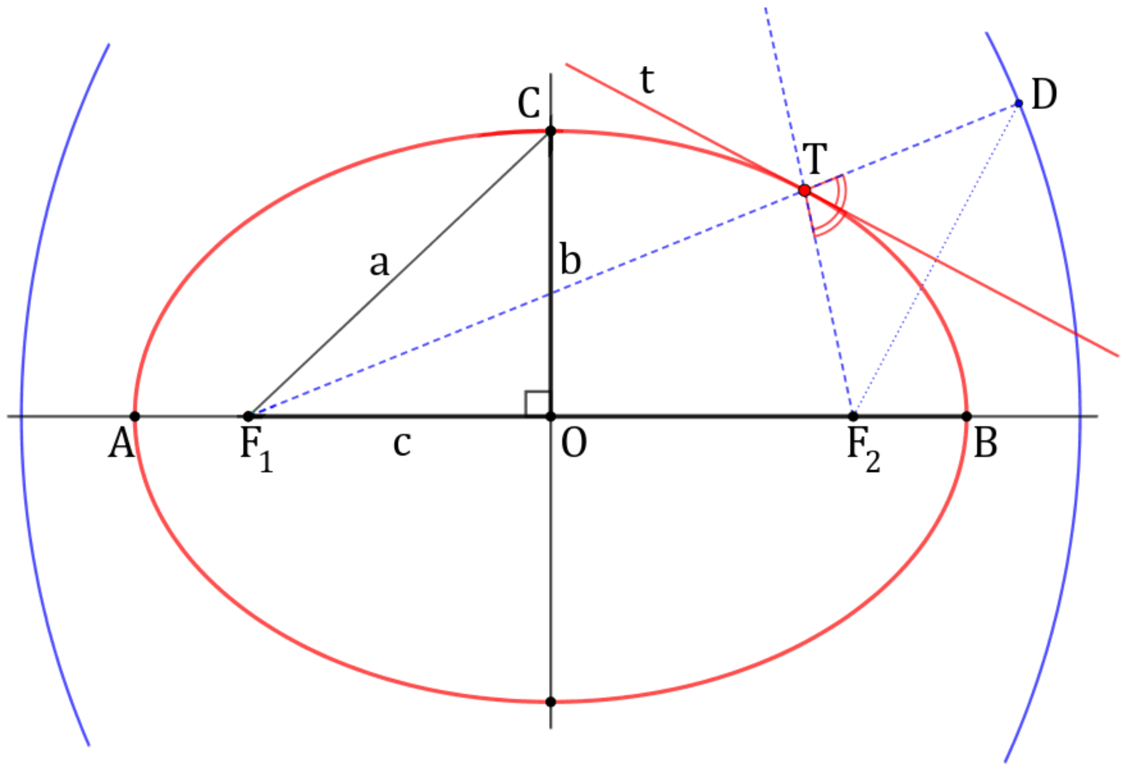
director circles:

center F_1, radius 2a
center F_2, radius 2a

focal radii (of point T):

lines F_1T and F_2T

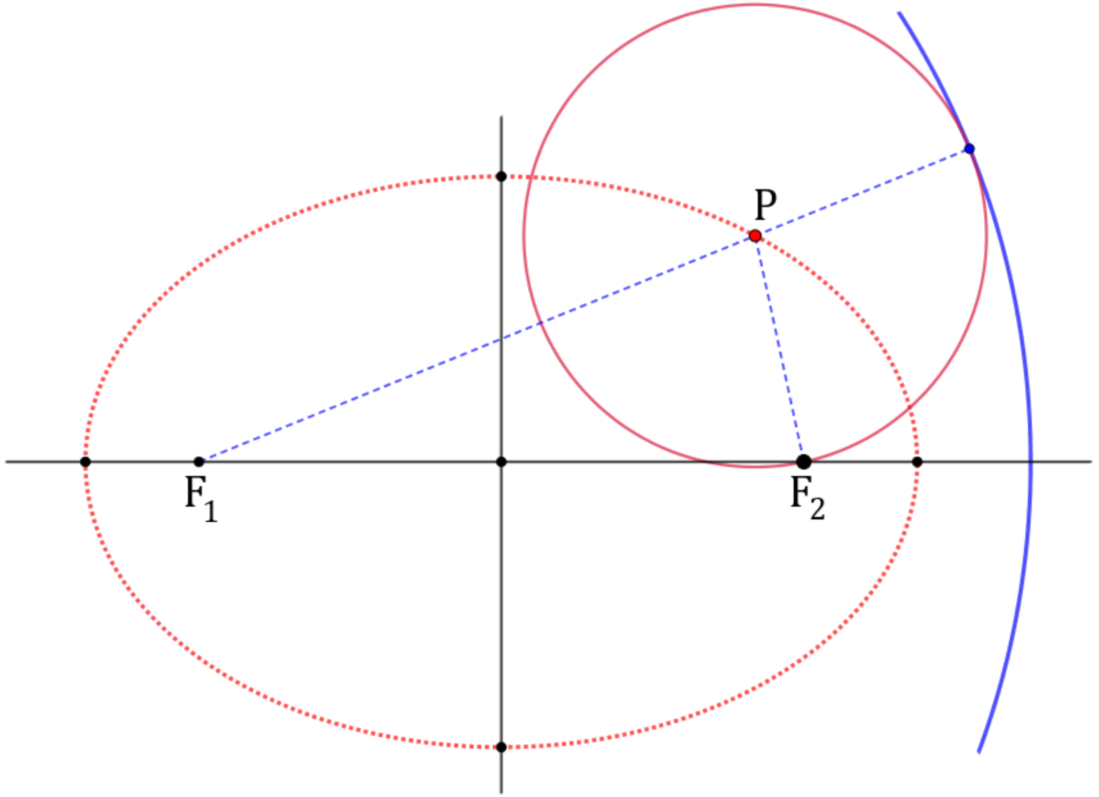
The angle bisector of the focal radii of point T is the tangent line of the ellipse at T.



Ellipse – definition with director circle

An ellipse is the locus of centers of circles so that these circles are tangents to a given circle and pass through a given point inside the given circle.

The given circle is one of the two director circles and the given point is a focus
(which is not the center of the chosen director circle).

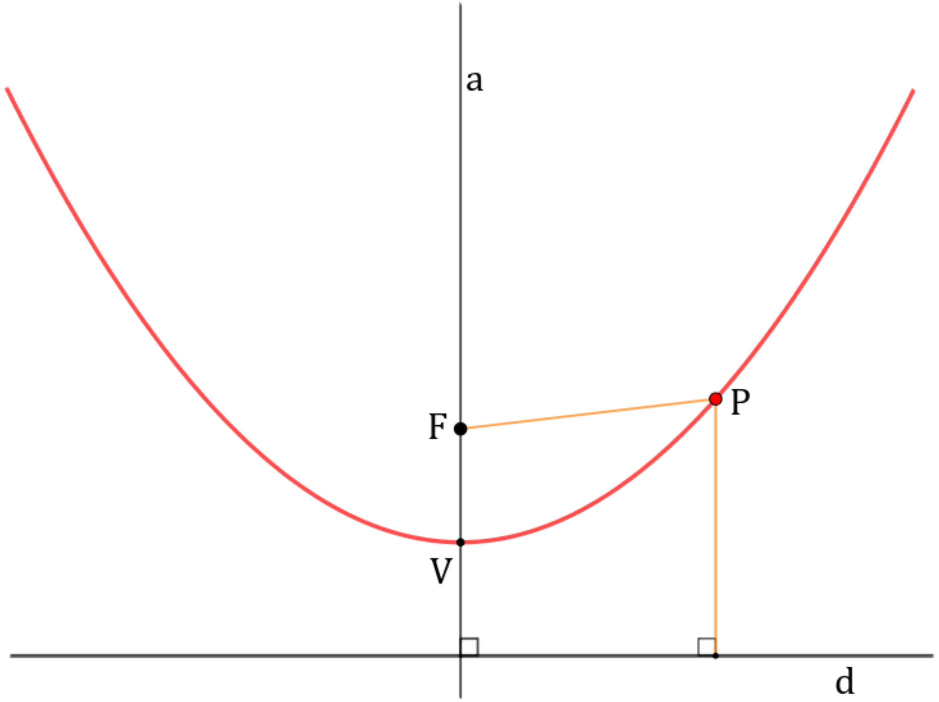


lecture

Parabola

Parabola: the locus of points in a plane with respect to a given point and a given line so that these points are equidistant from both given objects.

$$\boldsymbol{d}(\mathbf{P},\mathbf{F}) = \boldsymbol{d}(\mathbf{P},\mathbf{d})$$



F – focus

d – directrix

V – vertex

a – axis (of symmetry)

p = d(F,d) – parameter of the parabola (or – according to some books – p=d(F,V)=d(V,d))

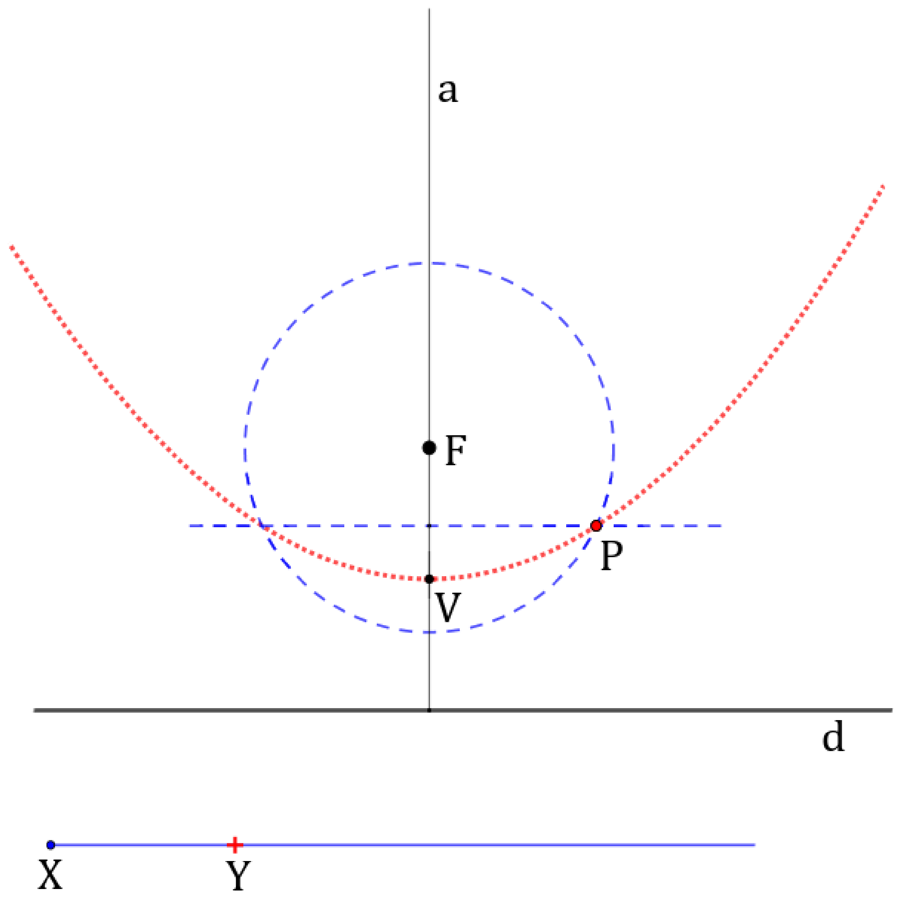
Parabola – construction of its points

The focus and the directrix are given.

Construct an arbitrary point of the parabola.

Using the definition,

1. XY is an arbitrary segment ($XY \geq \frac{p}{2}$)
2. drawing a circle with center F and radius XY, and a parallel line to d
   (the distance between parallel lines is XY)
3. the intersection points of the circle and the line are points of the parabola
   (d(P,F) = d(P,d))



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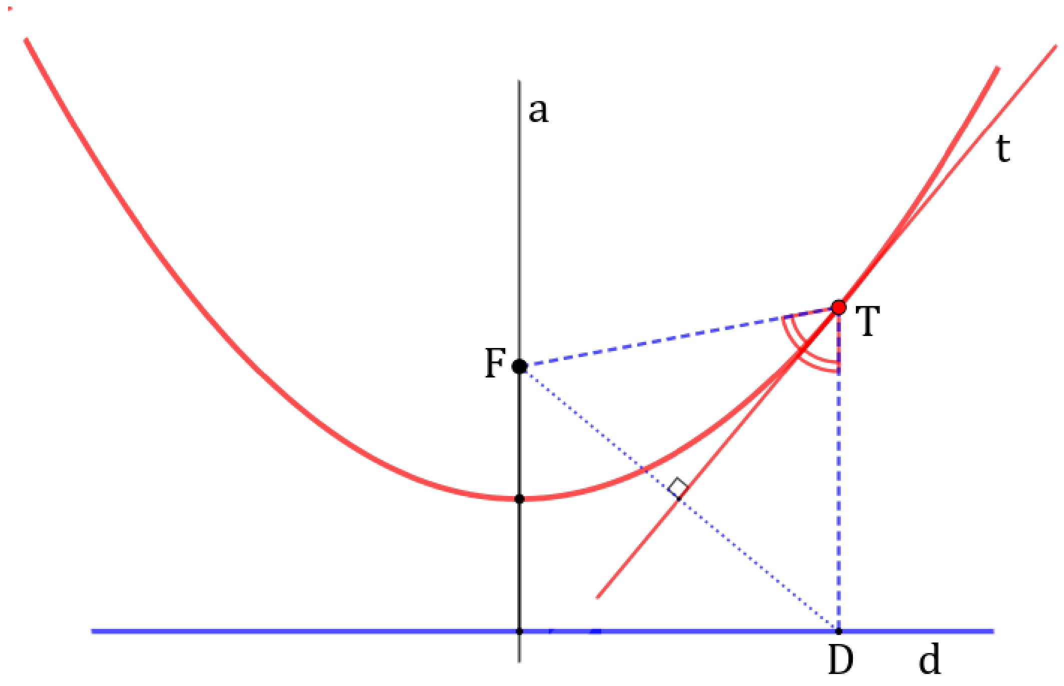
Parabola – further terminology

focal radii (of point T):

lines FT and DT

(where D is the foot of the perpendicular line to directrix d passing through T)

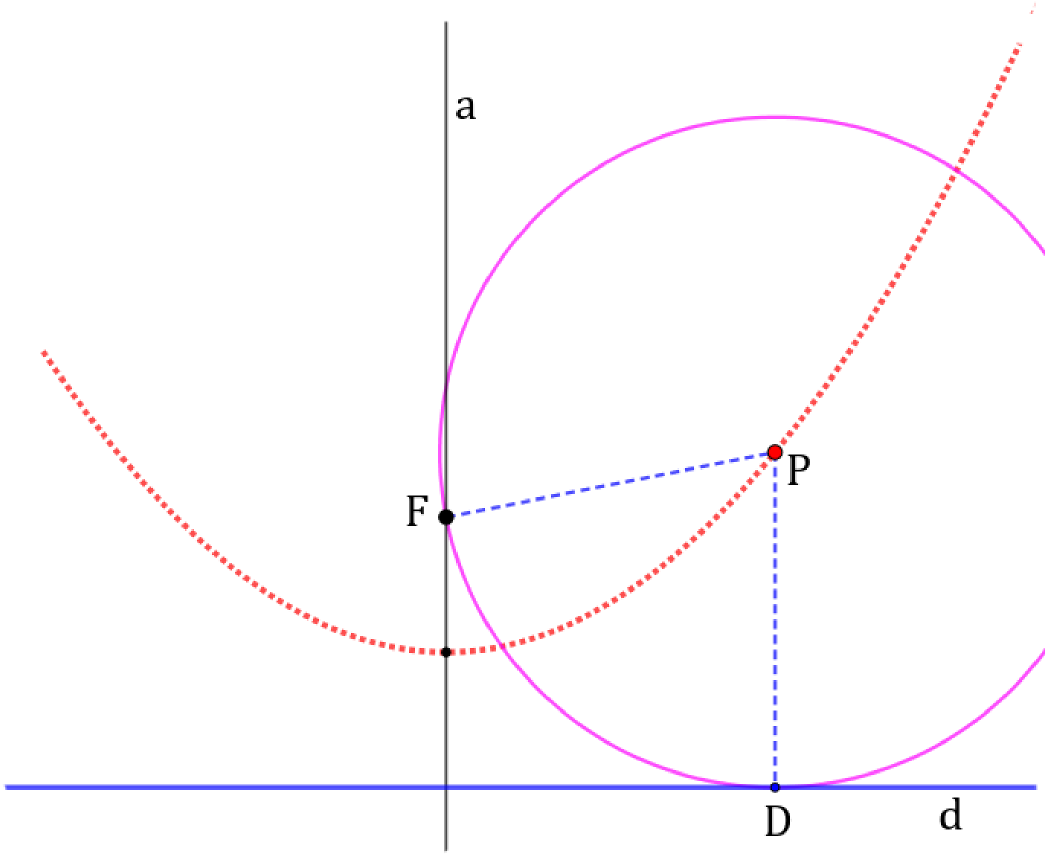
The angle bisector of the focal radii of point T is the tangent line of the parabola at T.



Parabola – definition with directrix

A parabola is the locus of centers of circles so that these circles are tangents to a given line and pass through a given point.

The given line is the directrix and the given point is the focus.

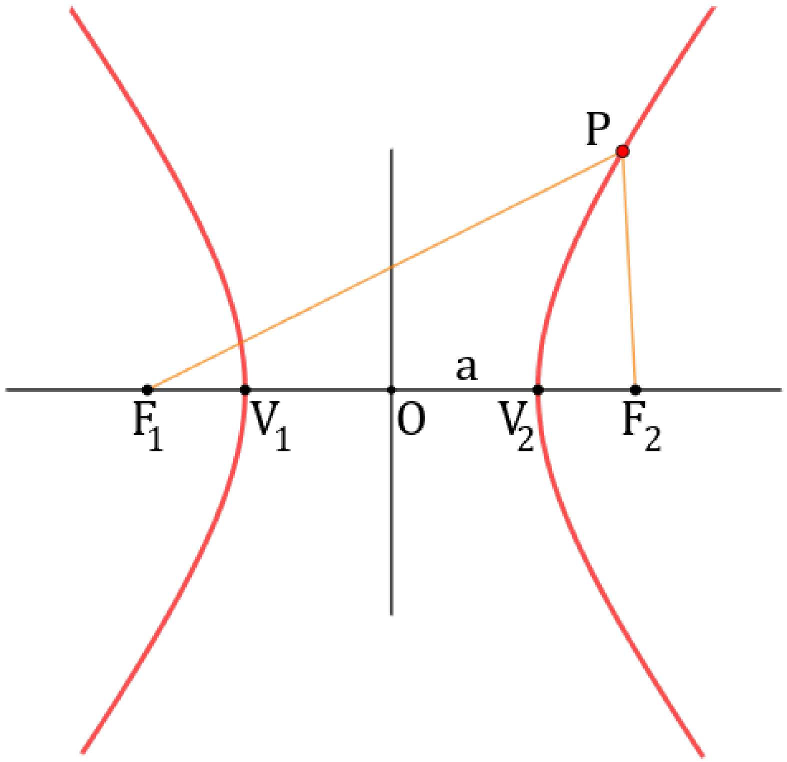


lecture

Hyperbola

Hyperbola: the locus of points so that
the absolute difference of the distances
to two given points is constant for any point P (of the set).

$$|\mathbf{F_1P - F_2P}| = \mathbf{constant} = 2a < F_1F_2$$



F_1, F_2 – *foci*
O – *center*
V_1, V_2 – *vertices*

$V_1V_2 = 2a$ – *major axis* (OV_1, OV_2 – semi-major axes)

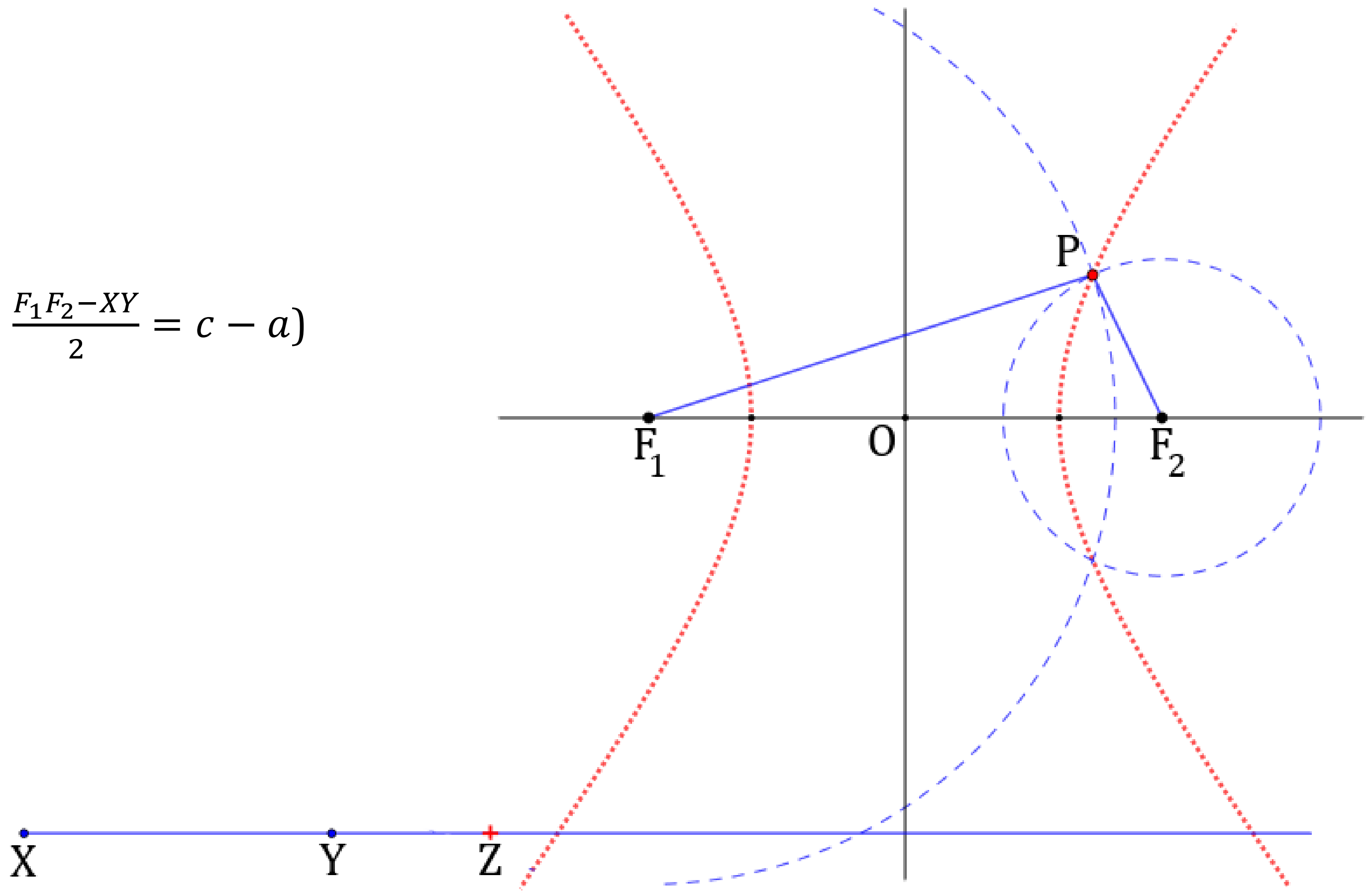
Hyperbola – construction of its points

The length of the major axis (2a) and two foci are given.

Construct an arbitrary point of the hyperbola.

Using the definition,

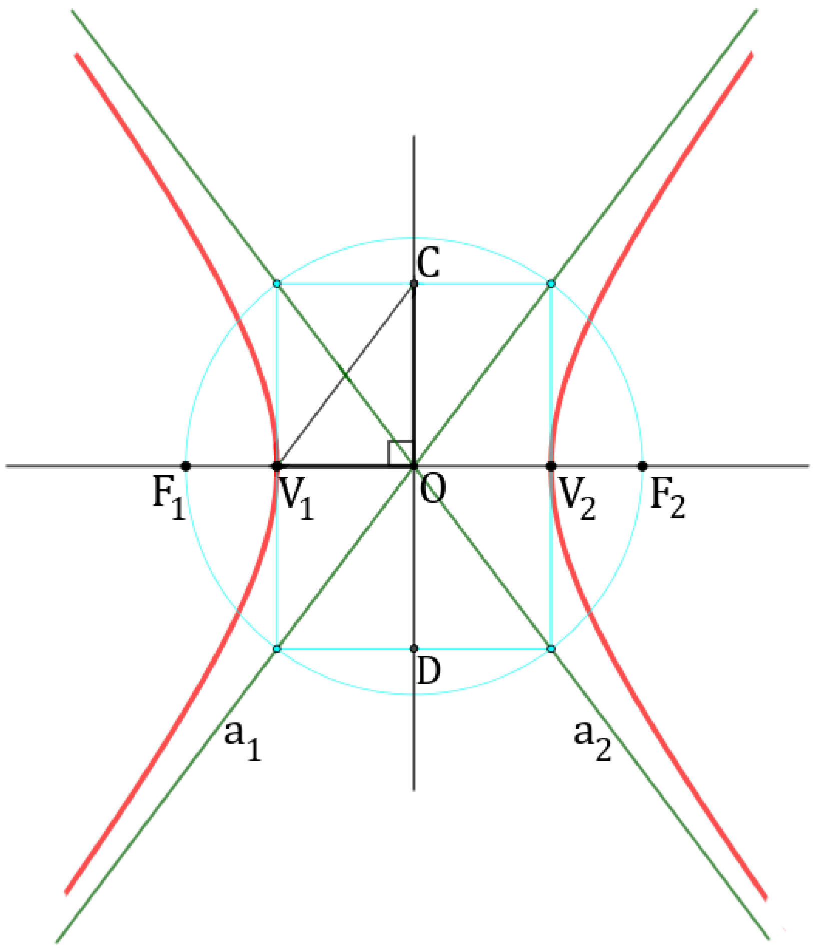
1. XY = 2a
2. Z is an arbitrary point on ray XY ($YZ \geq \frac{F_1F_2 - XY}{2} = c - a$)
3. drawing two circles:
   center F_1, radius XZ
   center F_2, radius YZ
4. the intersection points of circles
   are points of the hyperbola
   ($F_1P - F_2P = 2a$)



Hyperbola – asymptotes

OF_1, OF_2 – *linear eccentricity* (c)

1. Construct the circle with center O and radius c.
2. Draw perpendicular lines to major axis passing through V_1 and V_2.
3. Using intersection points of previous lines and circle, we obtain a rectangle (see light blue segments).



Diagonals of this rectangle are **asymptotes** of the hyperbola.

Asymptotes a_1 and a_2 never touch or intersect the hyperbola.

CD = 2b – *minor axis*

$a^2 + b^2 = c^2$

Hyperbola – further terminology

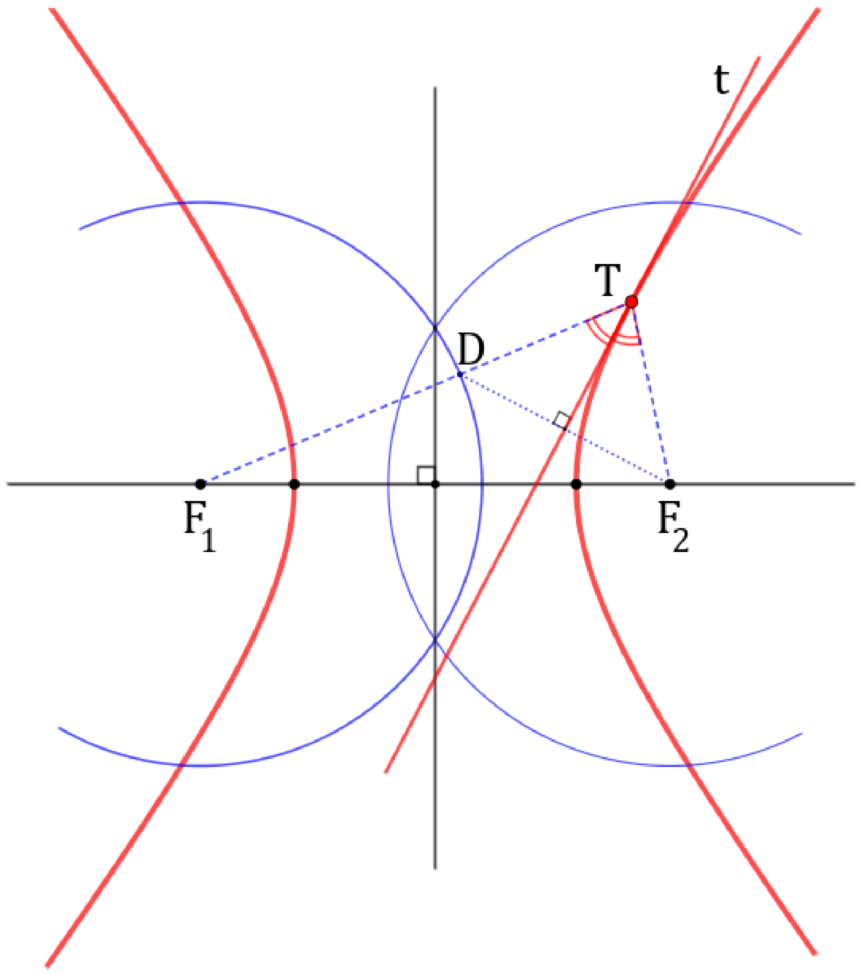
director circles:

center F_1, radius 2a
center F_2, radius 2a

focal radii (of point T):

lines F_1T and F_2T

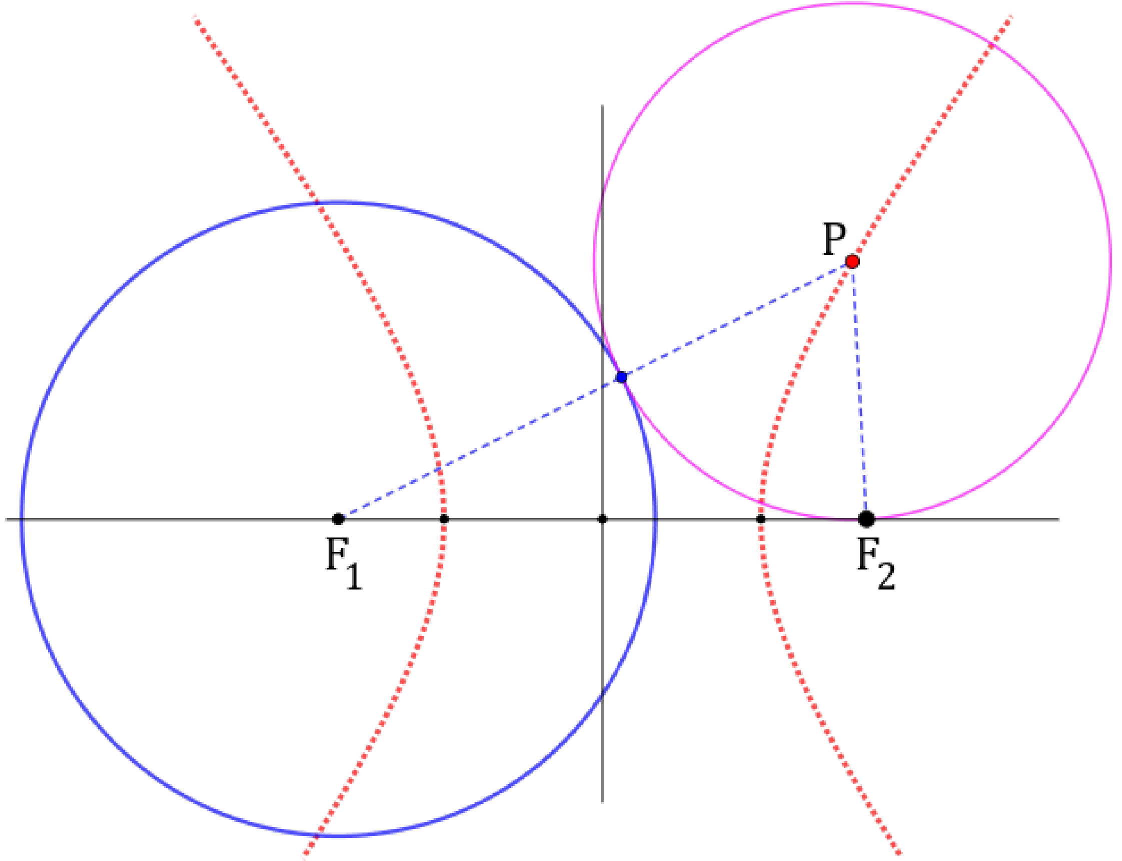
The angle bisector of the focal radii of point T is the tangent line of the hyperbola at T.



Hyperbola – definition with director circle

A hyperbola is the locus of centers of circles so that these circles are tangents to a given circle and pass through a given point outside the given circle.

The given circle is one of the two director circles and the given point is a focus
(which is not the center of the chosen director circle).



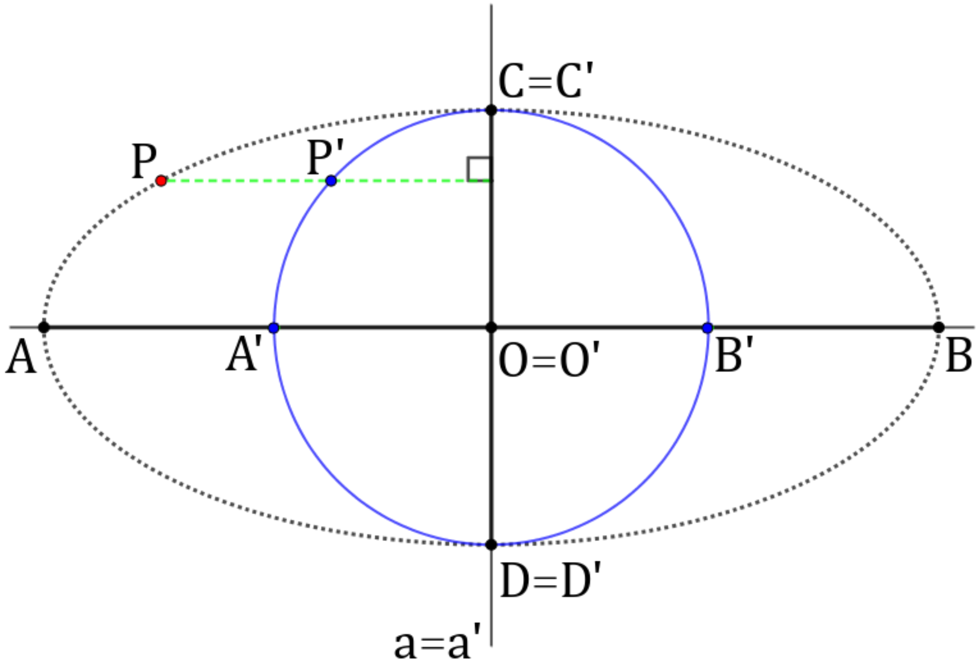
lecture

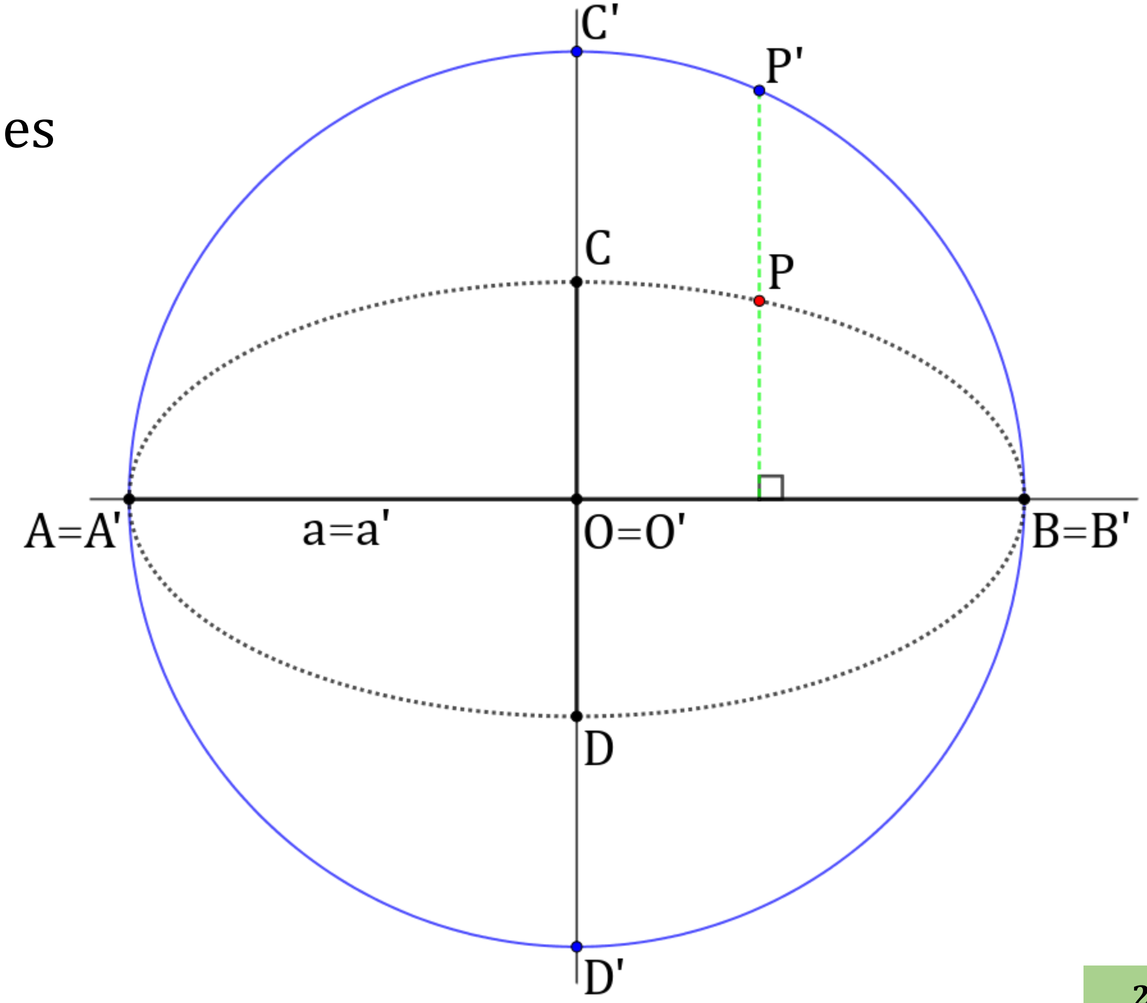
BASIC CONSTRUCTIONS ON ELLIPSES

An arbitrary affinity transforms an ellipse into an ellipse.

There are some special cases when **the affine image of an ellipse is a circle**.

If the axes of an ellipse are given, in two special cases
we can use *orthogonal affinity* to get
the image of an ellipse as a circle.

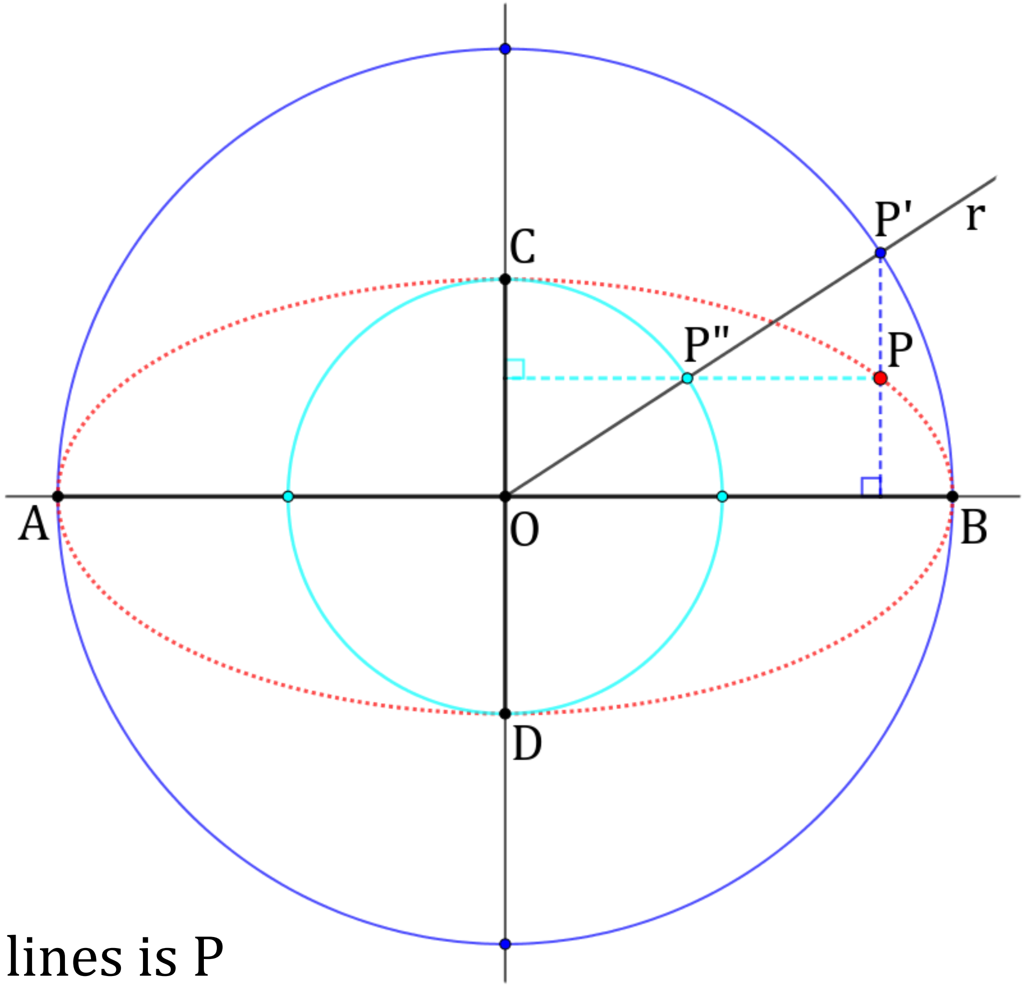




Two-circles construction of an ellipse

The major and minor axes of an ellipse are given.

Construct a new point of the ellipse.



Using both previous orthogonal affinities,

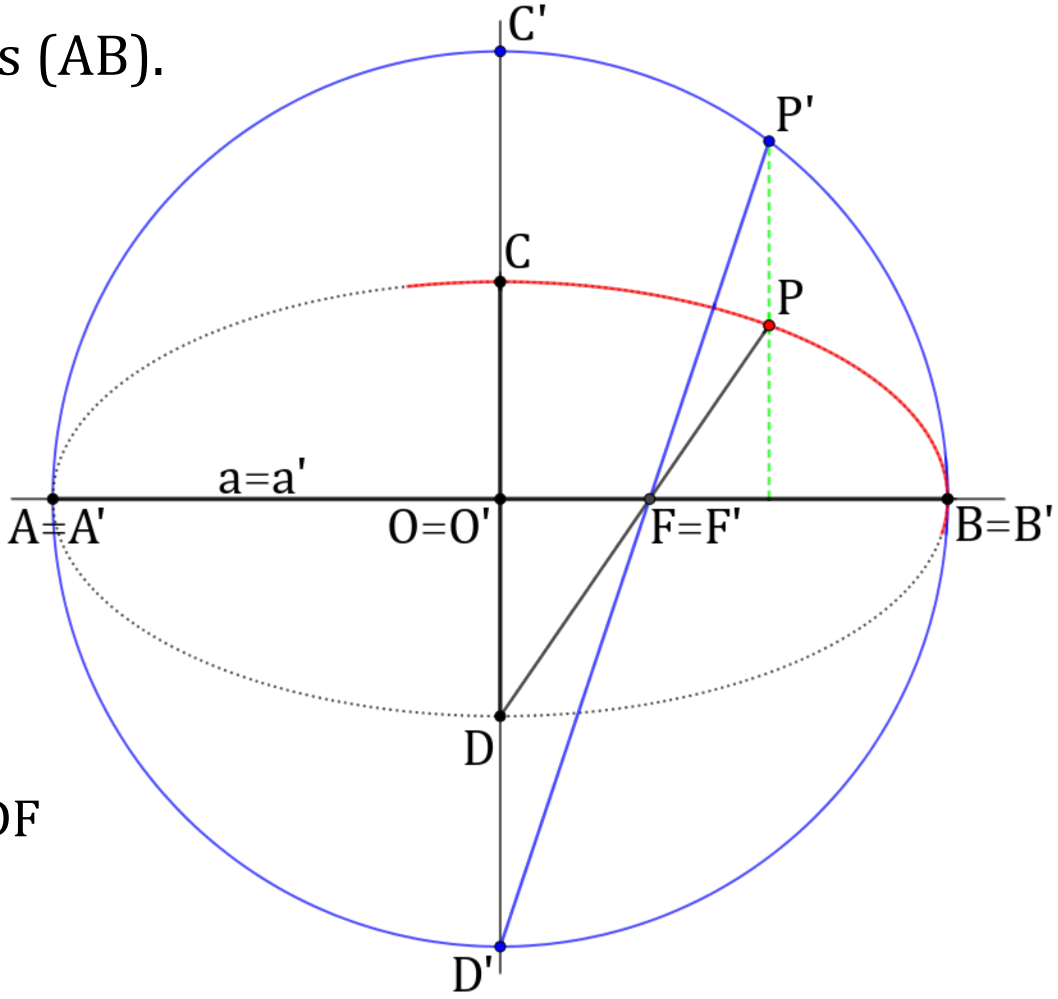
1. drawing two circles:
   center O, radius OA, and
   center O, radius OC
2. r is an arbitrary ray with vertex O
3. r intersects both circles: P’, P”
4. drawing perpendicular lines:
   to the major axis through P’, and
   to the minor axis through P”
5. the intersection point of the two perpendicular lines is P
   where P is a point of the ellipse

Construction of a new point of an ellipse

The major and minor axes of an ellipse are given. Construct a new point of the ellipse.

We use an orthogonal affinity whose axis is the major axis (AB).

1. If the axis of the affinity is AB, then O=O’, A=A’, and B=B’.
2. A diameter of the image circle is A’B’ so we can draw the image circle. (blue circle)
3. The direction of the affinity is perpendicular to the axis. It means that C’ and D’ lie on line CD (and the image circle as well.) → C’, D’
4. Choose an arbitrary point on the image circle: P’
   (P’ is in „the world of the image circle”. We must find point P which is in „the world of the ellipse”.)
5. Draw e.g. line D’P’. This line intersects the axis at point F=F’.
6. Line DF contains point P (we want to construct).
   (The image line of DF is D’F’=D’P’.)
7. Using the direction of the affinity, the intersection point of DF and the direction through P’ is point P.



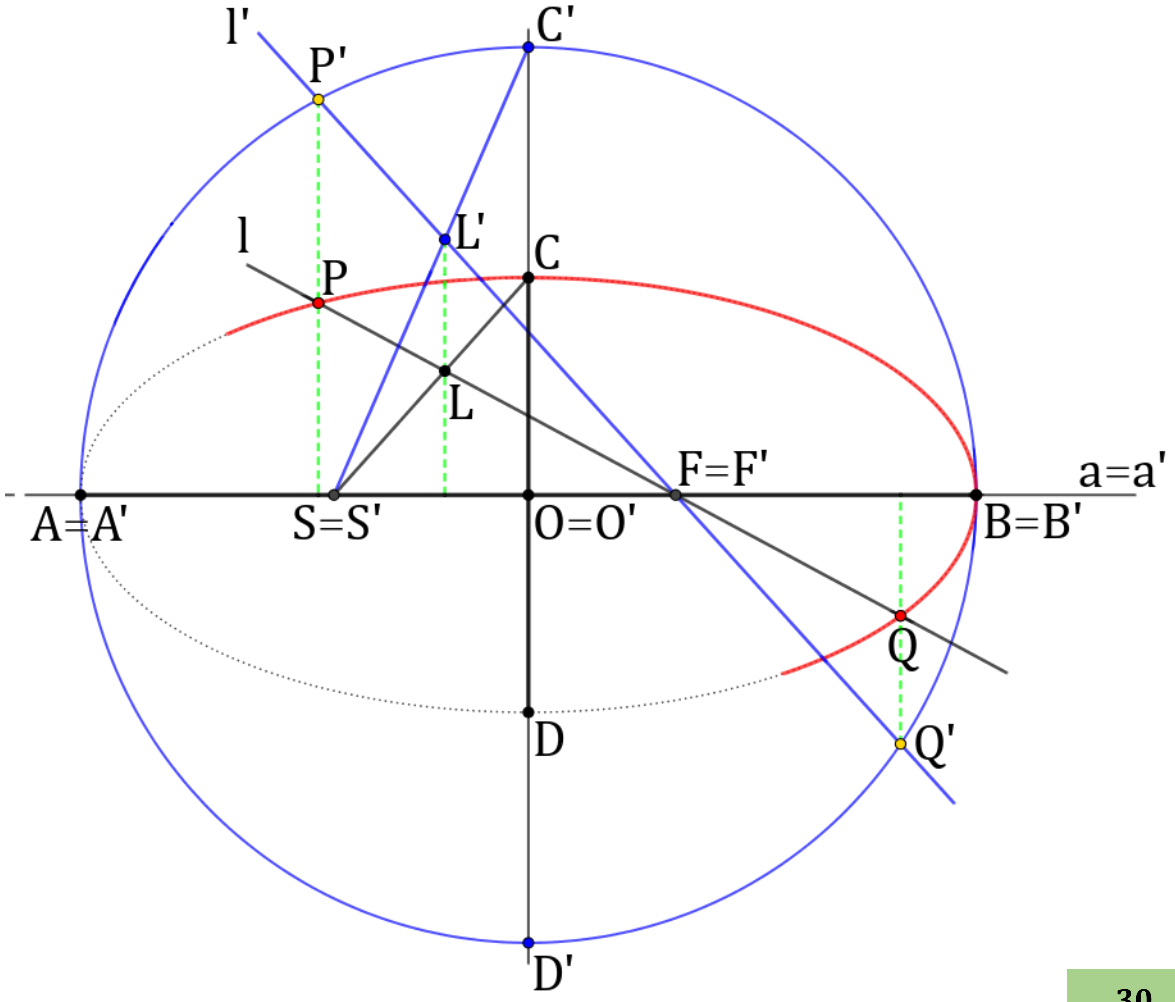
lecture

The intersection points of an ellipse and a line

The major and minor axes of an ellipse and a line are given. Construct their intersection points.

Using the orthogonal affinity with axis AB and a pair of point (C,C’) (where C’ is the intersection point of line CD and the circle with center O and radius OA),

1. L is an arbitrary point of line l
2. the fixed point of line LC is S=S’
   → the image of LC=SC is S’C’
3. using the direction of the affinity: L’ is on S’C’
4. the image of l is F’L’=l’ (F=F’ is the fixed point of l)
   (l is in „the world of the ellipse”, l’ is in „the world of the image circle”)
5. l’ intersects the image circle: P’, Q’
6. using the direction of the affinity:
   P, Q are the intersection points



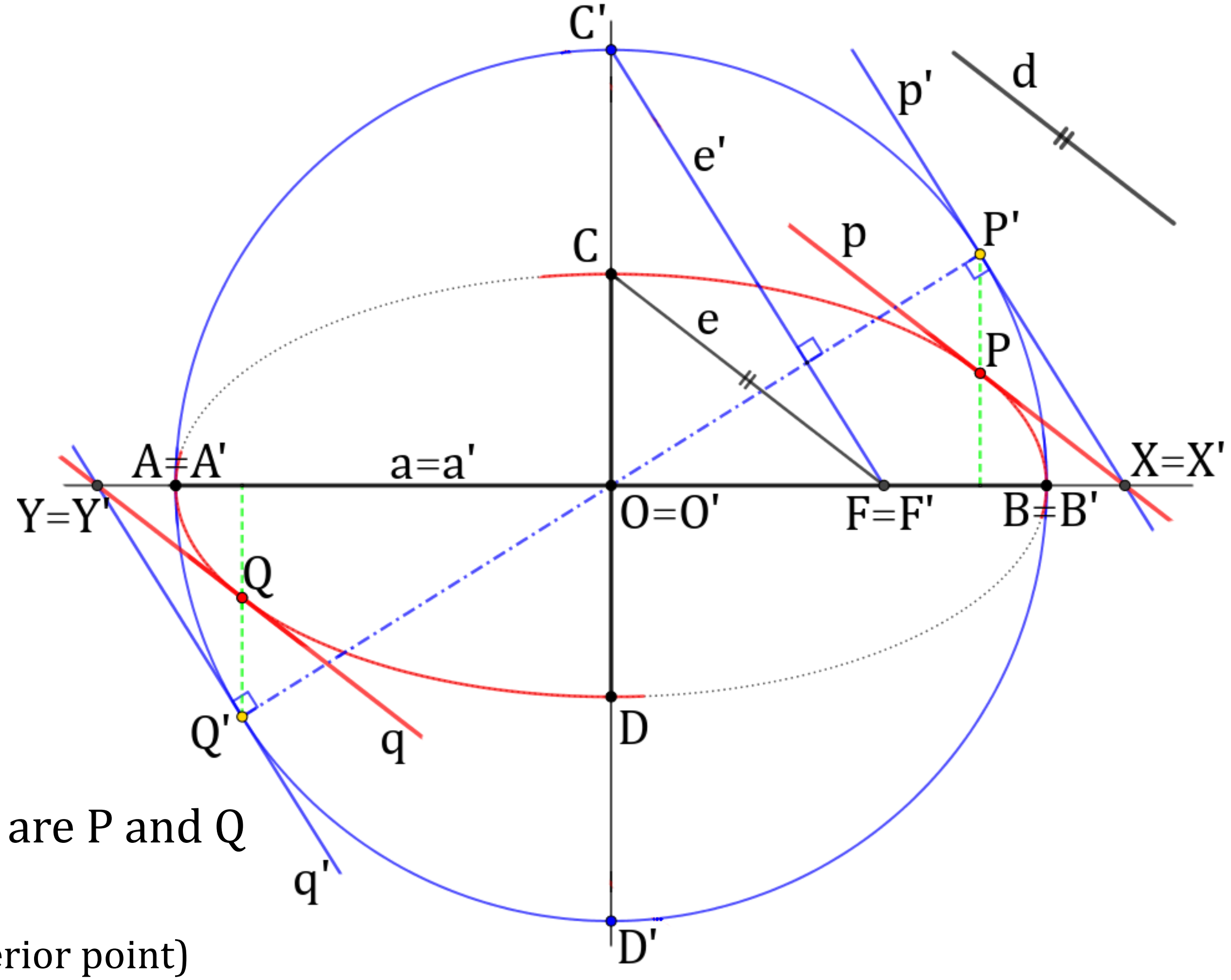
lecture

Tangent lines of an ellipse using a given direction

The major and minor axes of an ellipse and a line (d) are given. Construct the tangent lines of the given ellipse which are parallel to the given line.

Using the orthogonal affinity with axis AB and a pair of point (C,C’) (where C’ is the intersection point of line CD and the circle with center O and radius OA),

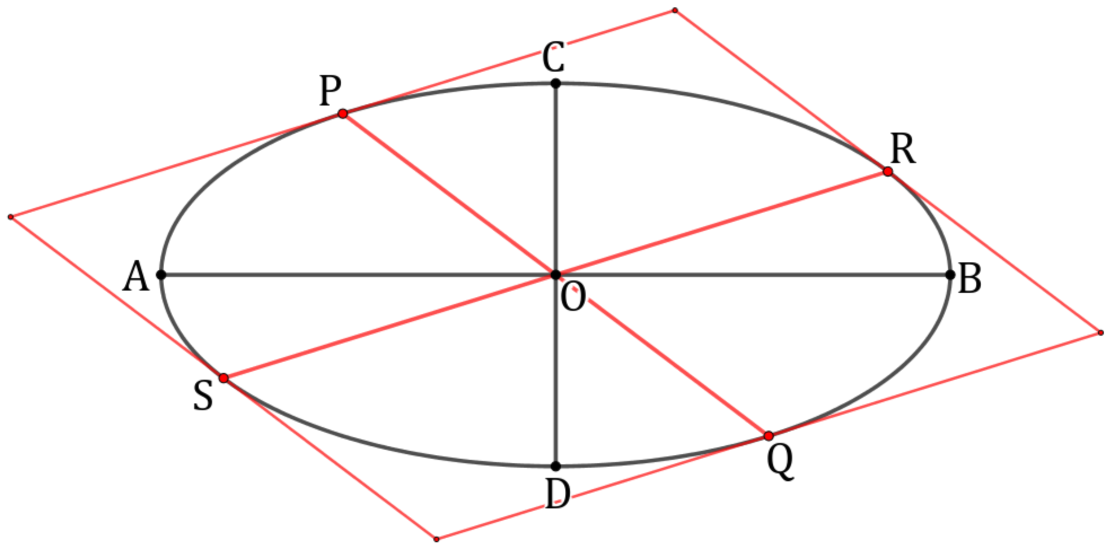
1. e is parallel to line d through point C
2. the image of e is C’F’=e’ (F=F’ is the fixed point of e)
3. constructing the tangent lines
   of the image circle which are parallel to e’
   → p’ with P’, and q’ with Q’
4. p passes through X=X’ and is parallel to d
   q passes through Y=Y’ and is parallel to d
5. using the directon of the affinity, points of tangency are P and Q



lecture / practical (tangent lines pass through an exterior point)

Conjugate diameters of an ellipse

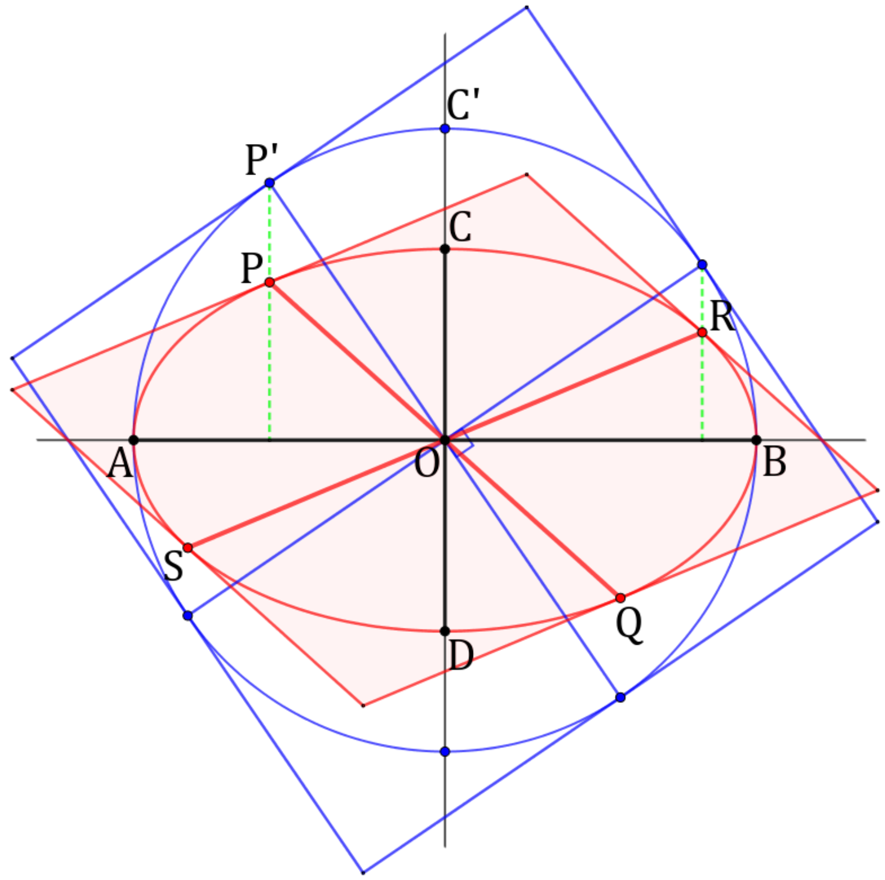
Conjugate diameters: two diameters of an ellipse so that the tangent lines of the ellipse at the endpoints of one diameter are parallel to the other diameter.



The tangent lines at the endpoints of conjugate diameters form a parallelogram.

Conjugate diameters bisect each other at center O.

Using an affinity, we can construct an arbitrary pair of conjugate diameters.



Conjugate diameters of a circle are perpendicular.

Construction of a new point of an ellipse
using its conjugate diameters

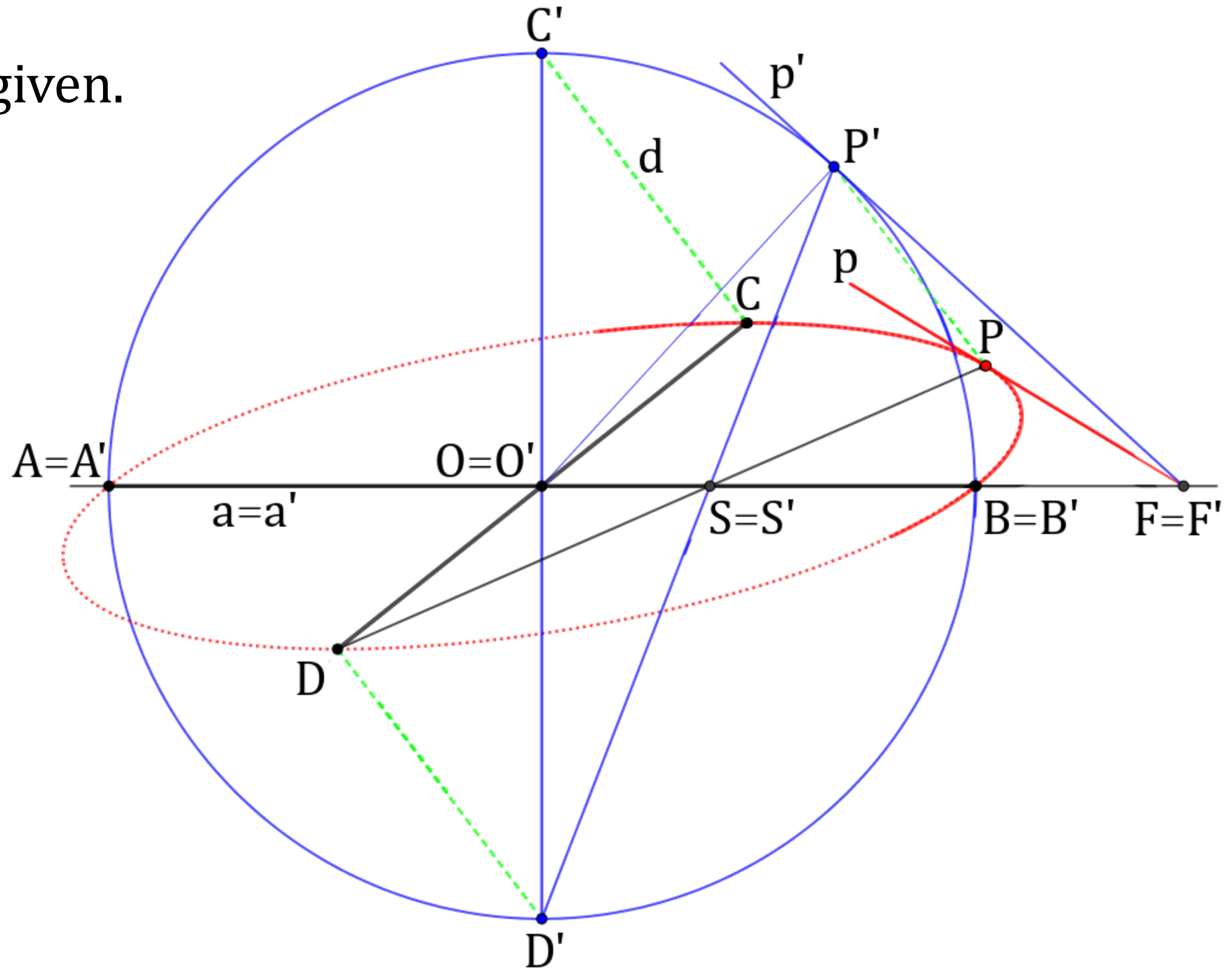
A pair of conjugate diamaters uniquely determine an ellipse.

A pair of conjugate diameters of an ellipse are given.
Construct a new point of the ellipse.

The main idea of the construction:

Using an axial affinity, if AB and CD are conjugate diameters of the given ellipse then A’B’ and C’D’ are perpendicular diameters of the image circle.

Consider the axial affinity with axis AB and direction CC’=d.



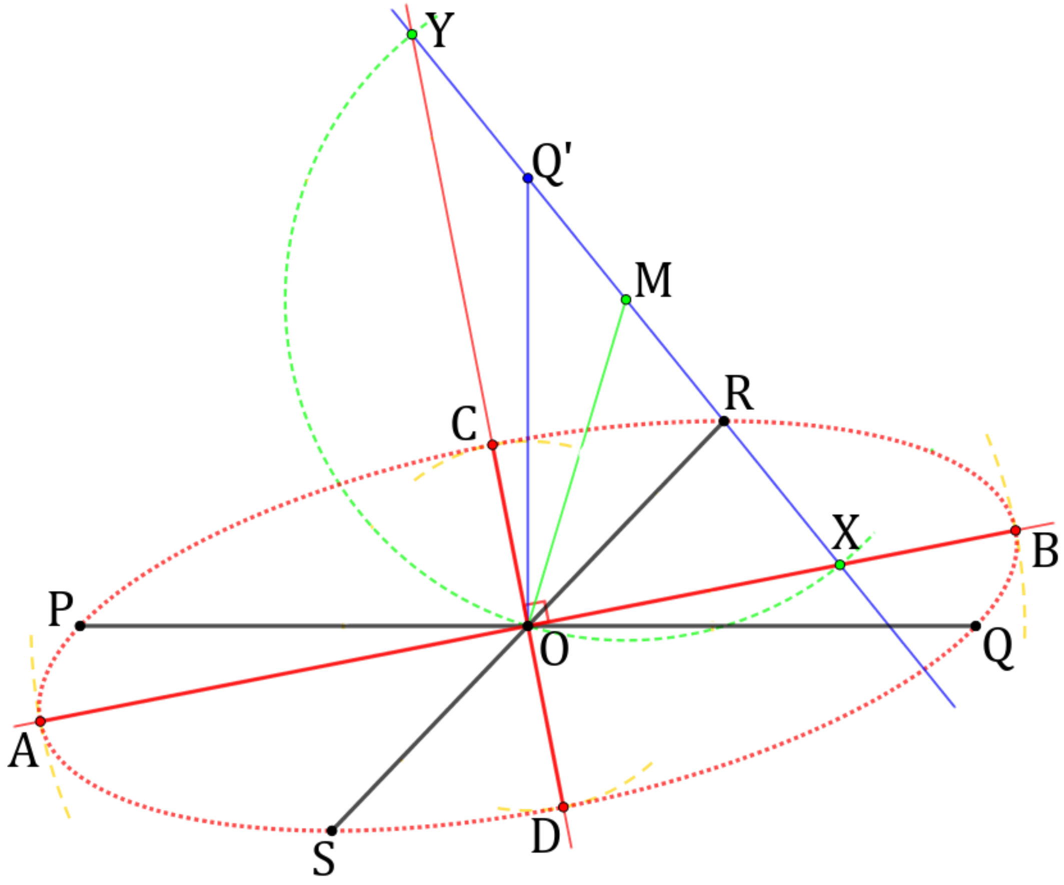
lecture and practical

Rytz's construction

Rytz's construction is useful when conjugate diameters of an ellipse are given and we would like to construct the axes of the given ellipse.

PQ and RS are conjugate diameters of an ellipse. Construct axis AB and CD.

1. Rotate e.g. OQ by 90°: OQ' .
2. M is the midpoint of RQ' and draw line RQ'.
3. Draw a circle with center M and radius MO.
4. This circle intersects line RQ' at points X and Y.
5. OX and OY are the lines of the axes.
6. The length of the major axis is Q'X , the length of the minor axis is Q'Y .

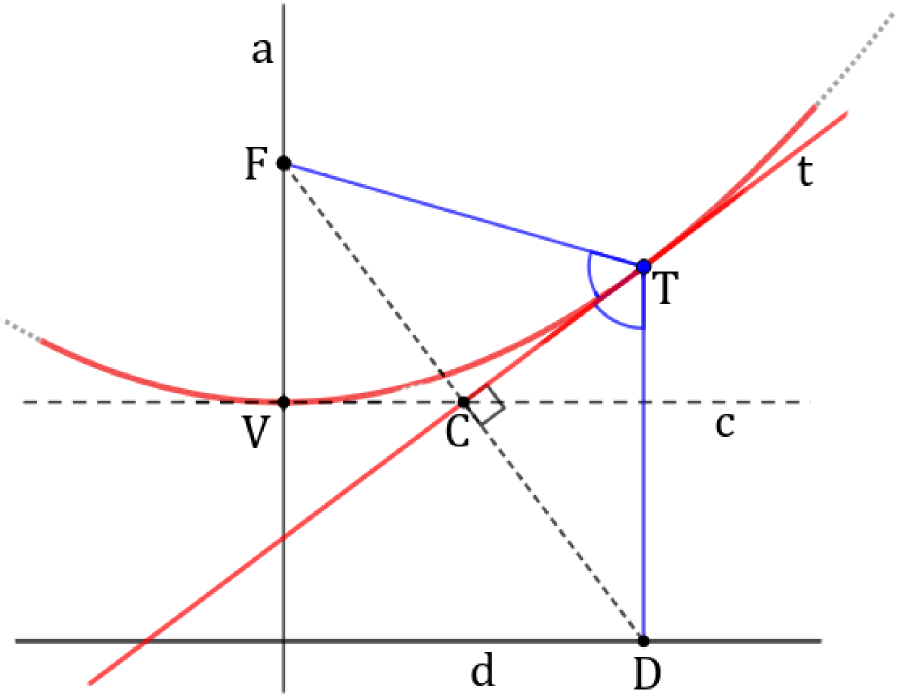


practical

BASIC CONSTRUCTIONS ON PARABOLAS

- F – focus, d – directrix
- TF = TD (See page 18)
- The angle bisector of the focal radii of point T is the tangent line of the parabola at T. (See page 20)

- Triangle TFD is an isosceles triangle.
- Tangent line c at vertex V is perpendicular to the axis.
- The midpoint of segment FD lies on line c.

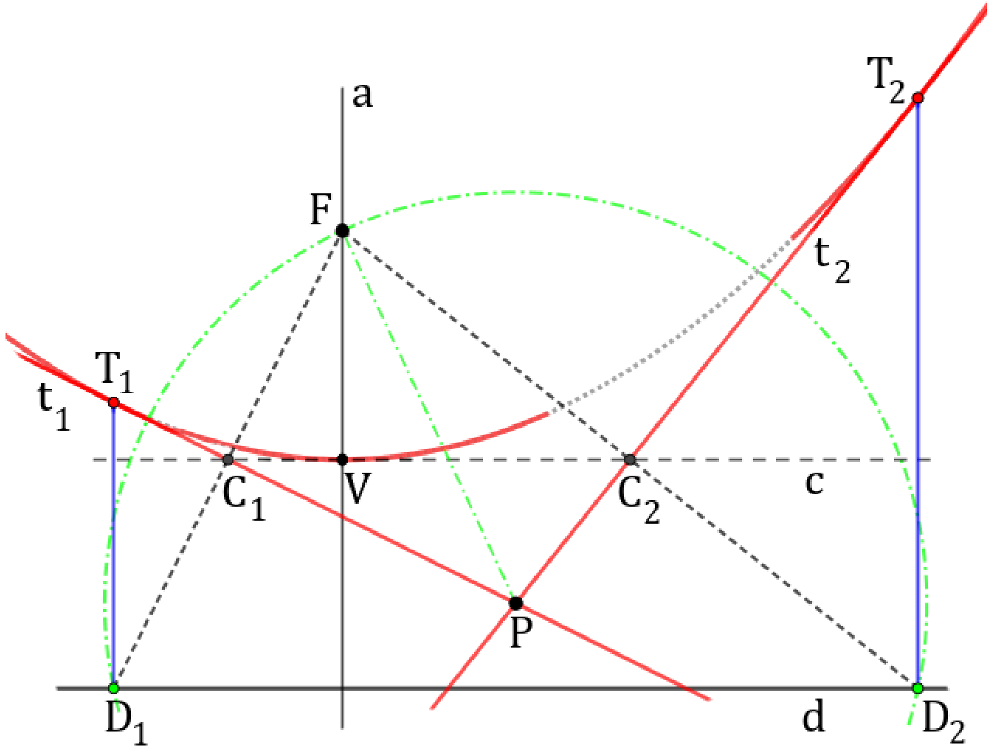


Construction of tangent lines to a parabola

The focus and the directrix of a parabola, and an exterior point (P) are given.
Construct the tangent lines of the given parabola which pass through the given point.

Steps of the construction

1. Draw a circle with center P and radius PF.
2. This circle intersects d at points D_1 and D_2.
3. The perpendicular bisector of segment FD_1 is t_1;
   the perpendicular bisector of segment FD_2 is t_2.
   These two lines are the tangent lines which pass through P.
4. Draw the focal radii which pass through D_1 and D_2 (perpendicular lines to d).
5. The intersection points of these focal radii and line t_1 and t_2 are the points of tangency: T_1 and T_2



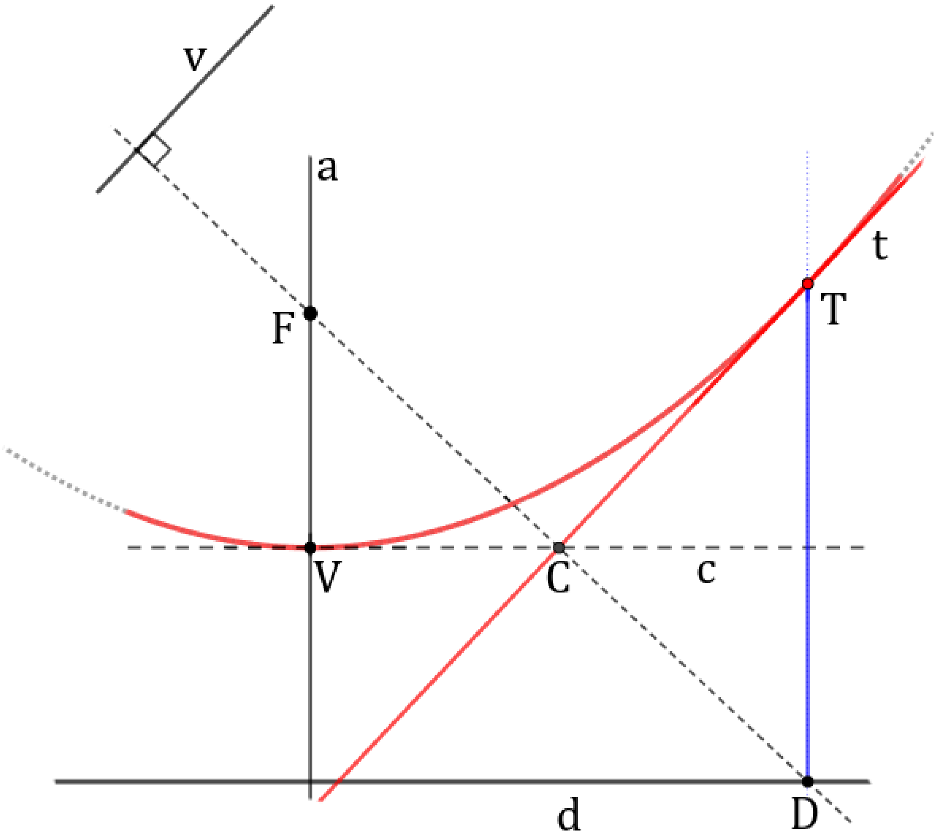
lecture

The tangent line of a parabola using a given direction

The focus and the directrix of a parabola, and a line (v) are given.
Construct the tangent line of the given parabola which is parallel to the given line.

Steps of the construction

1. Draw a perpendicular line to v which passes through F.
2. The intersection point of this line and d is point D.
3. The perpendicular bisector of FD is line t (the tangent line).
4. Draw the focal radius which passes through D (perpendicular line to d).
5. This line intersects line t at point T (where T is the point of tangency).



lecture

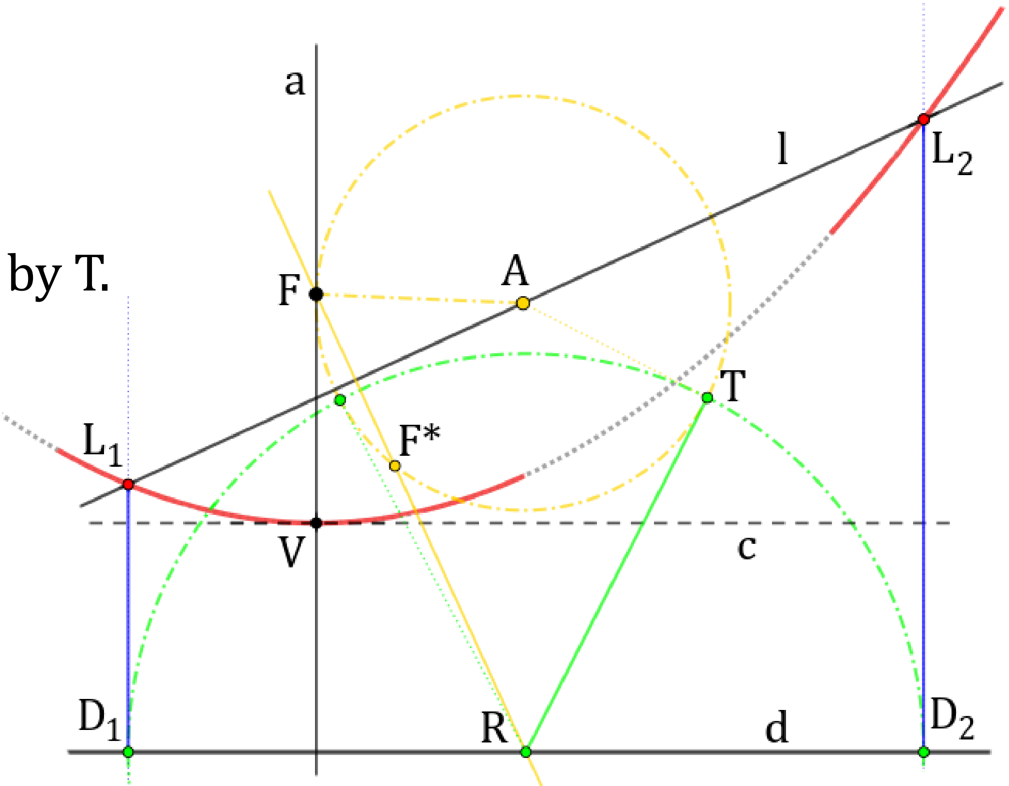
The intersection points of a parabola and a line

The focus and the directrix of a parabola, and line l are given. Construct their intersection points.

Steps of the construction

1. Reflect point F using line l as the axis: F*. Draw line FF* as well. (~radical line)
2. Line FF* intersects line d at point R.
3. Draw an arbitrary circle with center A and radius AF.
   (R lies on the radical line of all „yellow circles".)
4. Construct a tangent line to the arbitrary circle which passes through R. The point of tangency is denoted by T.
5. Draw a circle with center R and radius RT.
6. This circle intersects d at points D_1 and D_2.
7. Draw the two focal radii which pass D_1 and D_2.
   (perpendicular lines to d)
8. These focal radii intersect line l at L_1 and L_2.

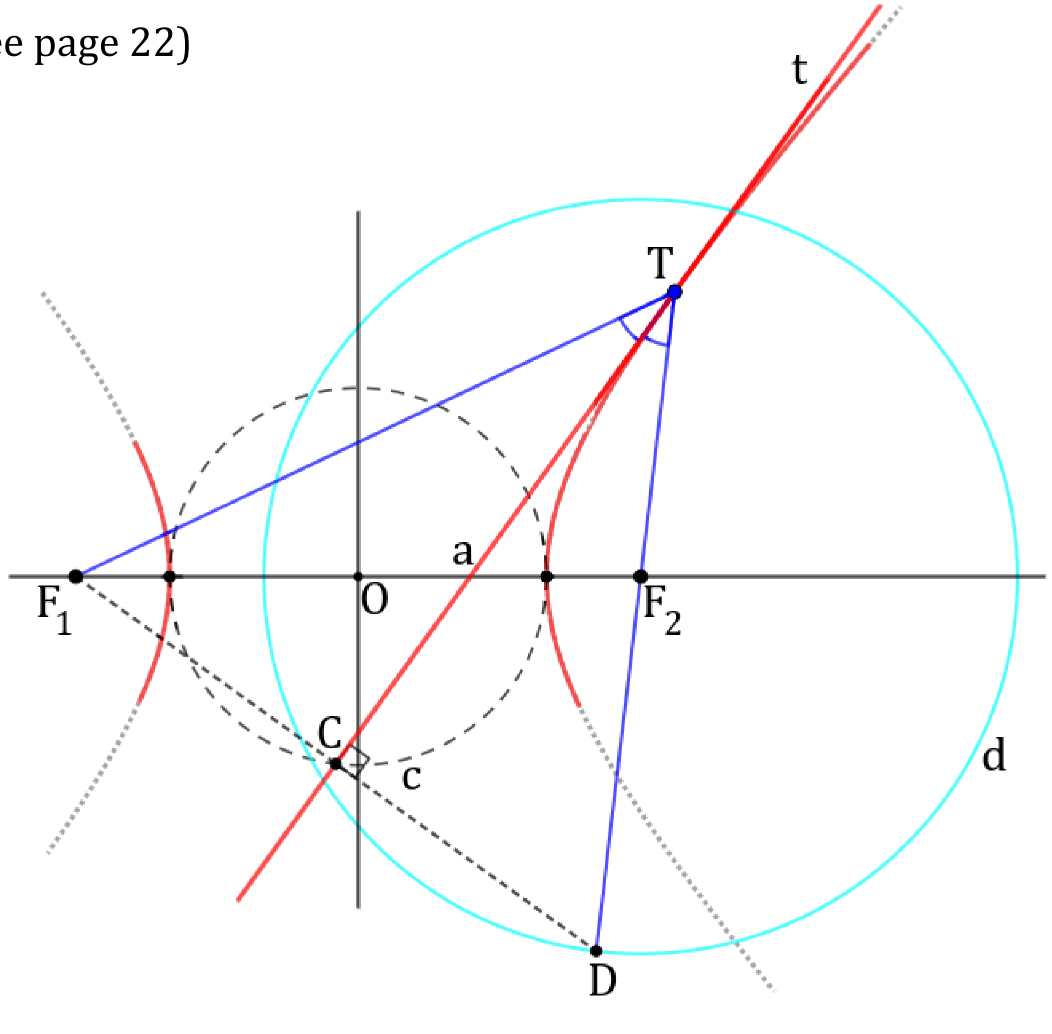
(This construction uses a special definiton of parabolas – see page 21.)



lecture

BASIC CONSTRUCTIONS ON HYPERBOLAS

- The length of the segment of the two vertices is 2a. (See page 22)
- $F_1T - F_2T = 2a$ (See page 22)
- c – circle with center O and radius a
  d – director circle (with center F_2 and radius 2a)
- The angle bisector of the focal radii of point T
  is the tangent line of the hyperbola at T. (See page 25)



- Triangle TF_1D is an isosceles triangle.
- The focal radius TF_2 intersects d at point D.
- The midpoint of segment F_1D lies on circle c.

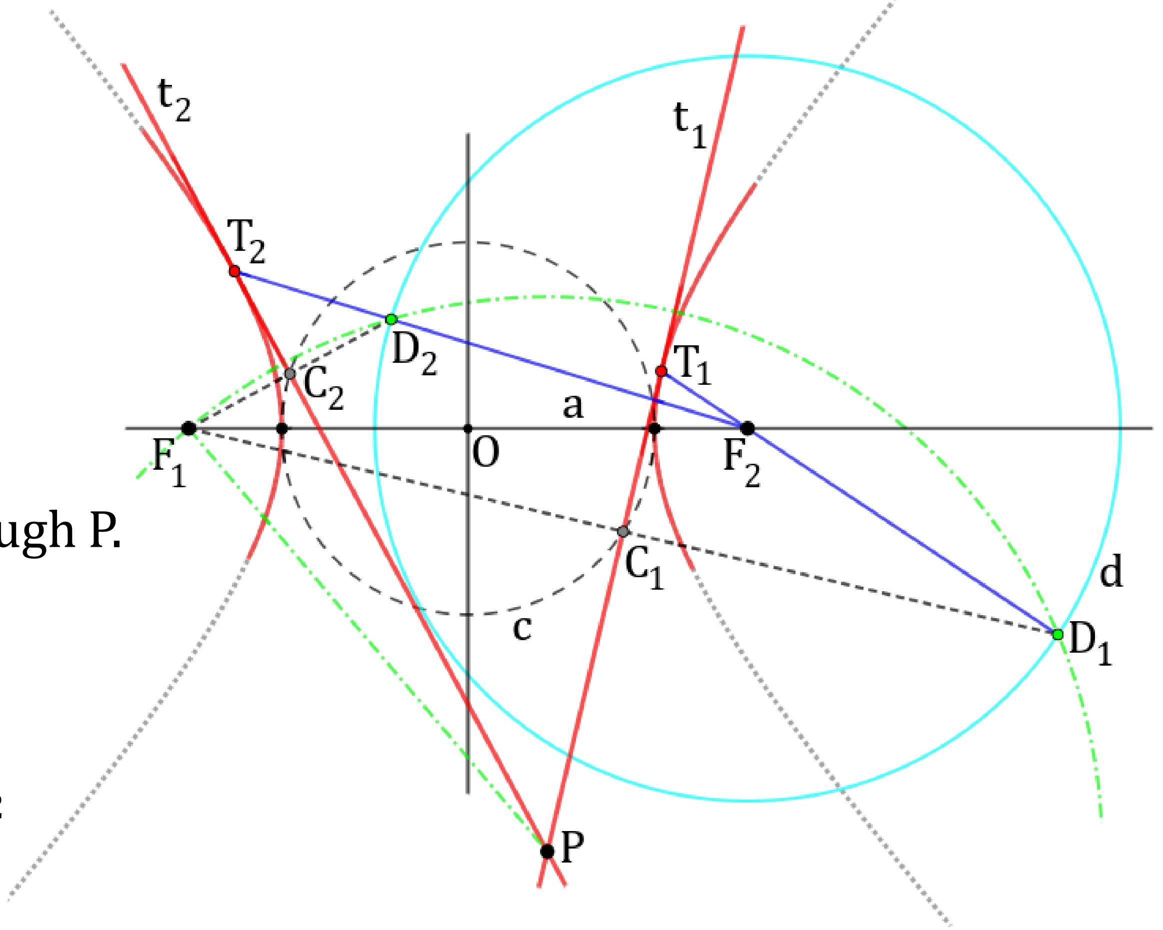
Construction of tangent lines to a hyperbola

The foci and the major axis of a hyperbola, and an exterior point (P) are given.
Construct the tangent lines of the given hyperbola which pass through the given point.

Steps of the construction

Draw circle c and a director circle (d, center F_2).

1. Draw a circle with center P and radius PF_1.
2. This circle intersects d at points D_1 and D_2.
3. The perpendicular bisector of segment F_1D_1 is t_1; the perpendicular bisector of segment F_1D_2 is t_2. These two lines are the tangent lines which pass through P.
4. Draw the focal radii which pass through D_1 and D_2 (lines F_2D_1 and F_2D_2).
5. The intersection points of these focal radii and line t_1 and t_2 are the points of tangency: T_1 and T_2



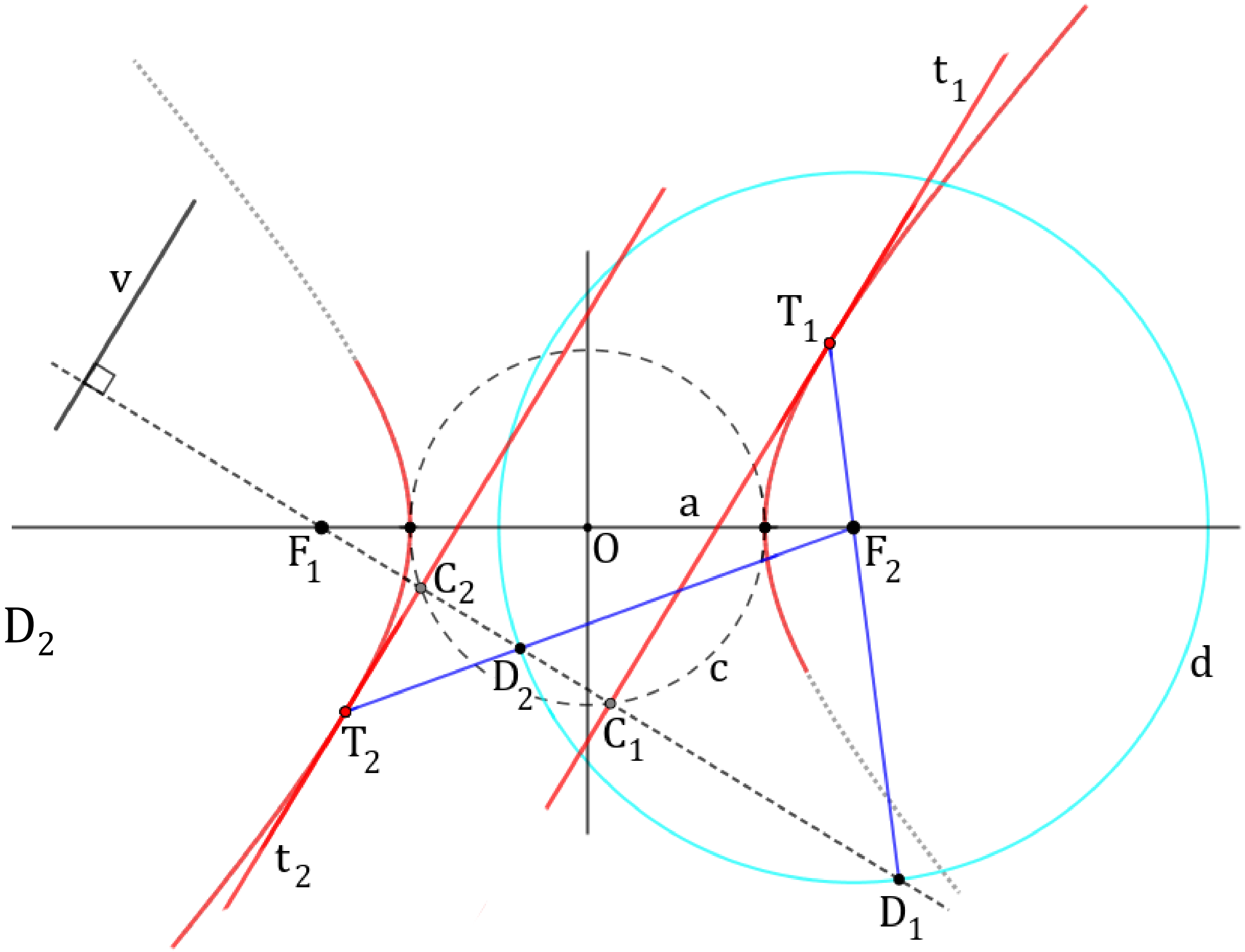
Tangent lines of a hyperbola using a given direction

The foci and the major axis of a hyperbola, and a line (v) are given.
Construct the tangent lines of the given hyperbola which are parallel to the given line.

Steps of the construction

Draw circle c and a director circle (d, center F_2).

1. Draw a perpendicular line to v which passes through F_1.
2. The intersection points of this line and circle d are points D_1 and D_2.
3. The perpendicular bisectors of F_1D_1 and F_1D_2 are lines t_1 and t_2 (the tangent lines).
4. Draw the focal radii which pass through D_1 and D_2 (lines F_2D_1 and F_2D_2).
5. These lines intersect lines t_1 and t_2 at points T_1 and T_2 (where T_1 and T_2 are the points of tangency).



The intersection points of a hyperbola and a line

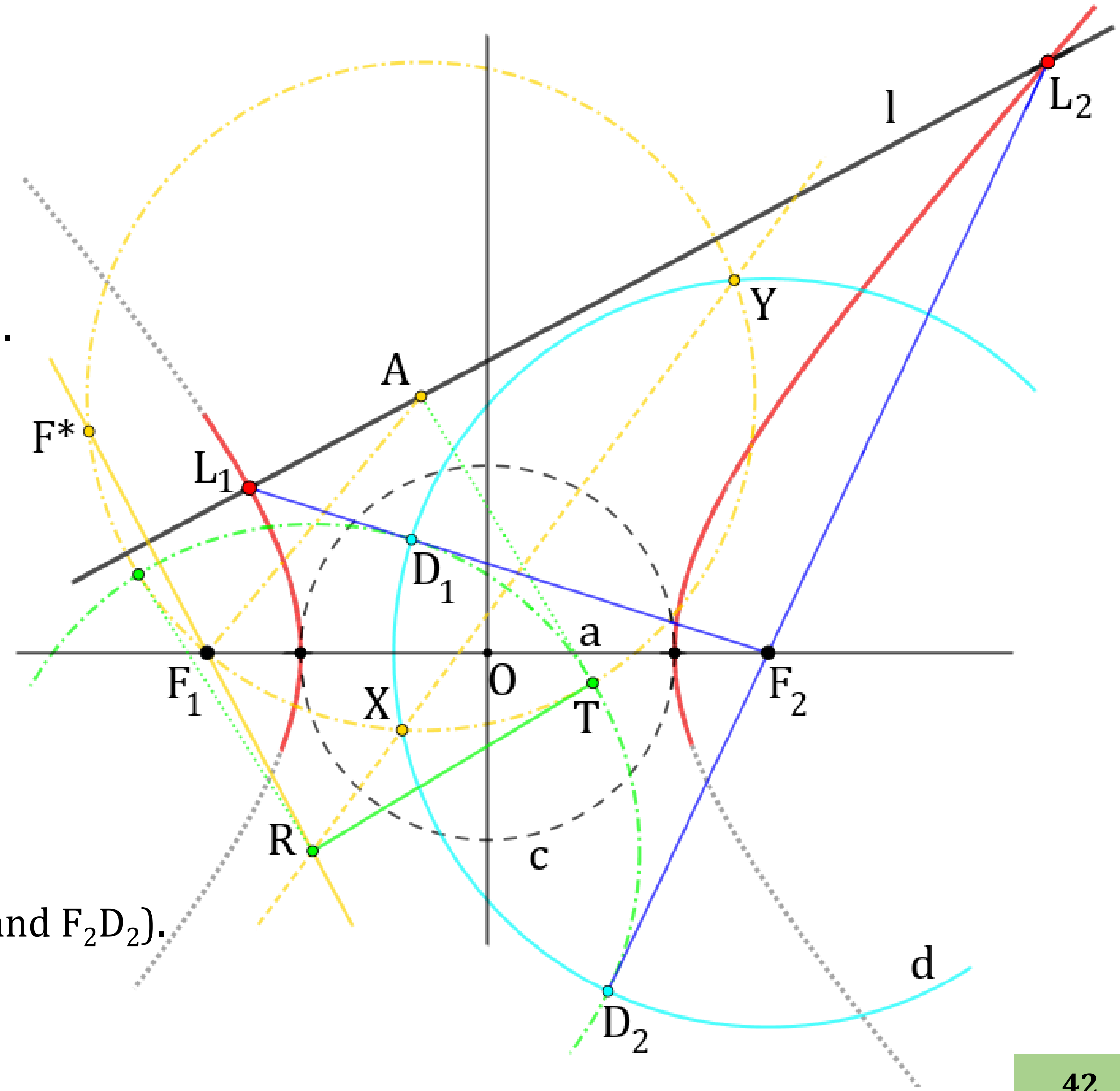
The foci and the major axis of a hyperbola, and line l are given. Construct their intersection points.

Steps of the construction

Draw a director circle (d, center F_2).

1. Reflect point F_1 using line l as the axis: F*. Draw line F_1F^*.
2. Draw an arbitrary circle with center A and radius AF_1.
3. This circle intersects circle d at X and Y. Draw line XY.
4. The intersection point of F_1F^* and XY is point R.
   (FF* is the radical line of all „yellow circles”; R is the radical point.)
5. Construct a tangent line to the arbitrary circle which passes through R. The point of tangency is T.
6. Draw a circle with center R and radius RT.
7. This circle intersects d at points D_1 and D_2.
8. Draw the two focal radii which pass D_1 and D_2 (lines F_2D_1 and F_2D_2).
9. These focal radii intersect line l at L_1 and L_2.

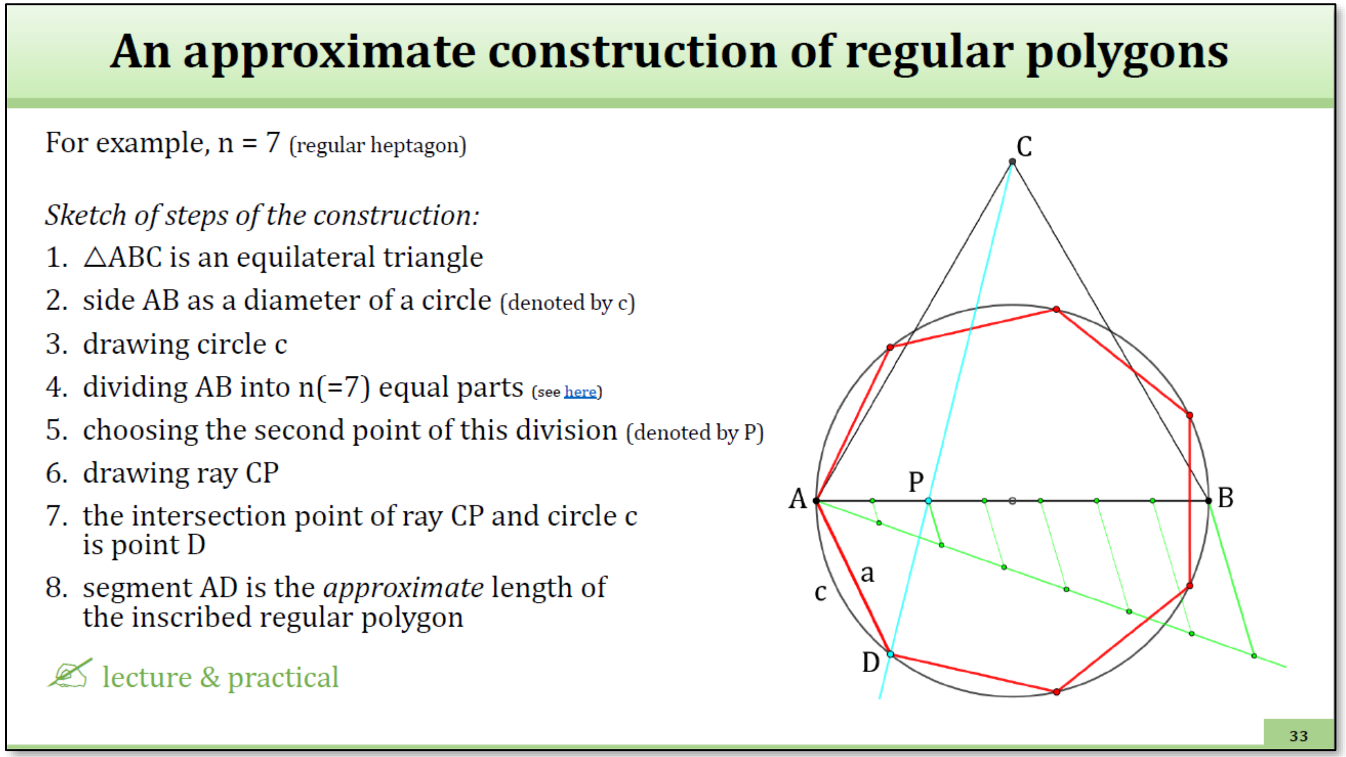
(This construction uses a special definiton of hyperbolas – see page 26.)

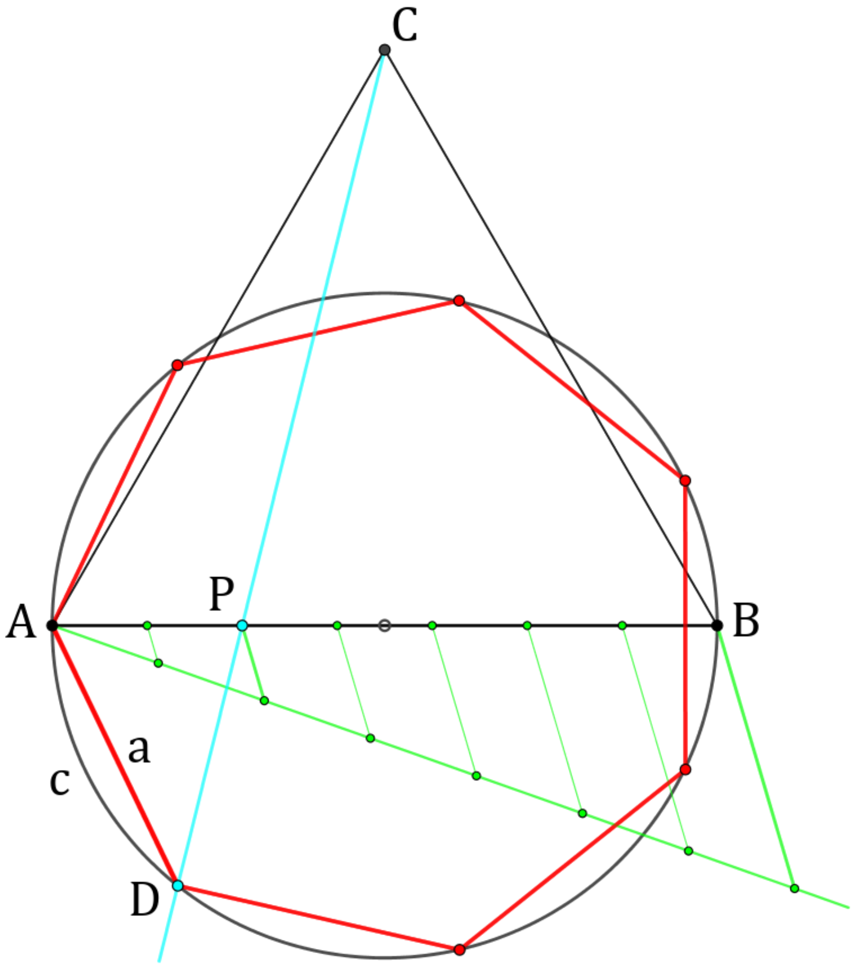


APPROXIMATE CONSTRUCTIONS

An approximate construction of regular polygons

→ See Geometrical Constructions 1



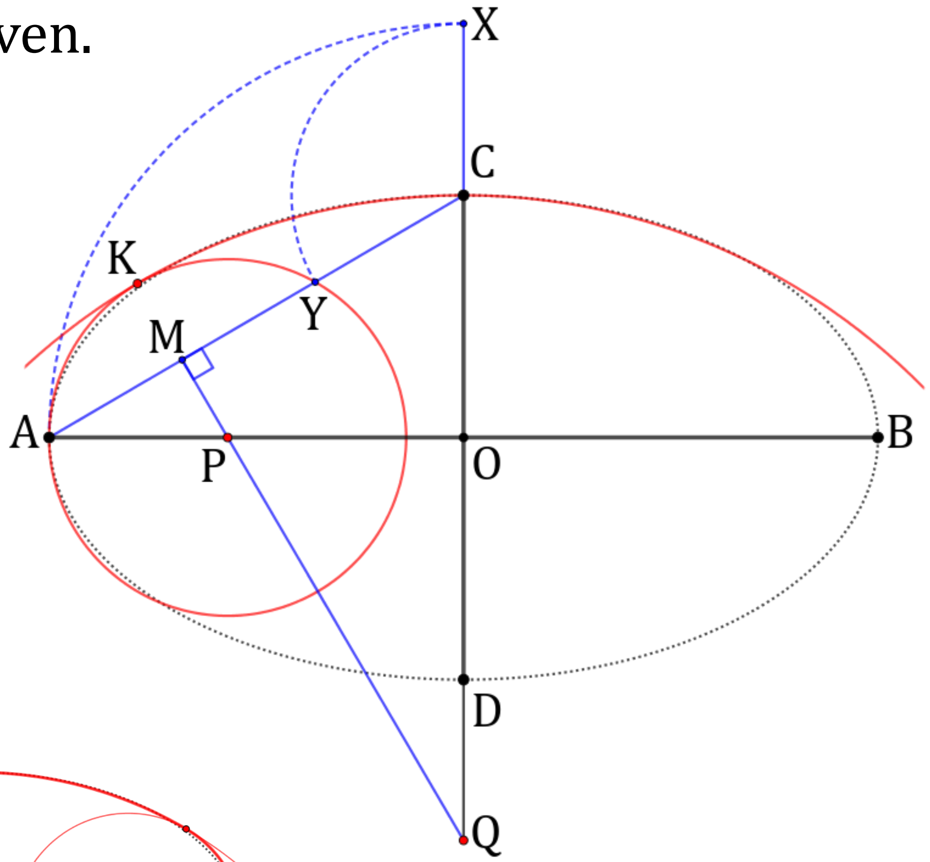


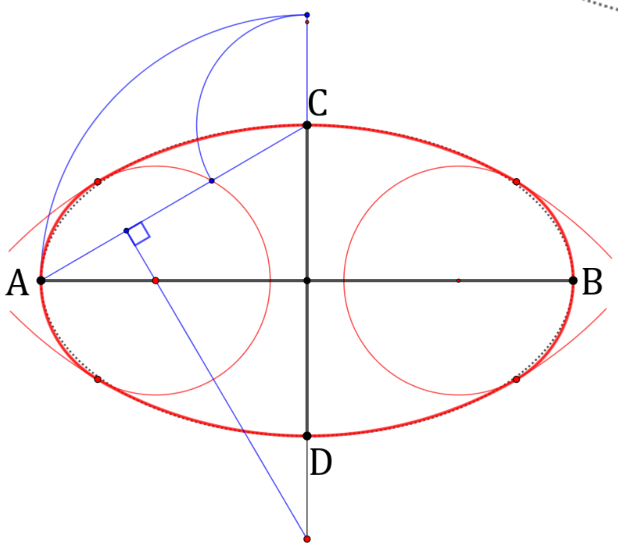
Four-center method of the construction of an ellipse

The major axis AB and the minor axis CD of an ellipse are given. Approximate an ellipse with using four circles.

Steps of the construction

1. Copy the semi-major axis on ray OC. $\Longrightarrow$ OA=OX
2. Draw segment AC.
3. Draw a circle with center C and radius CX.
4. This circle intersects AC at point Y.
5. Construct the perpendicular bisector of AY. (The midpoint is M.)
6. The bisector intersects the axes at P and Q.
7. Draw a circle with center P and radius PA. Draw a circle with center Q and radius QC.
8. The point of tangency of the circles is K.
9. Arcs AK and KC approximate a quarter of the given ellipse.





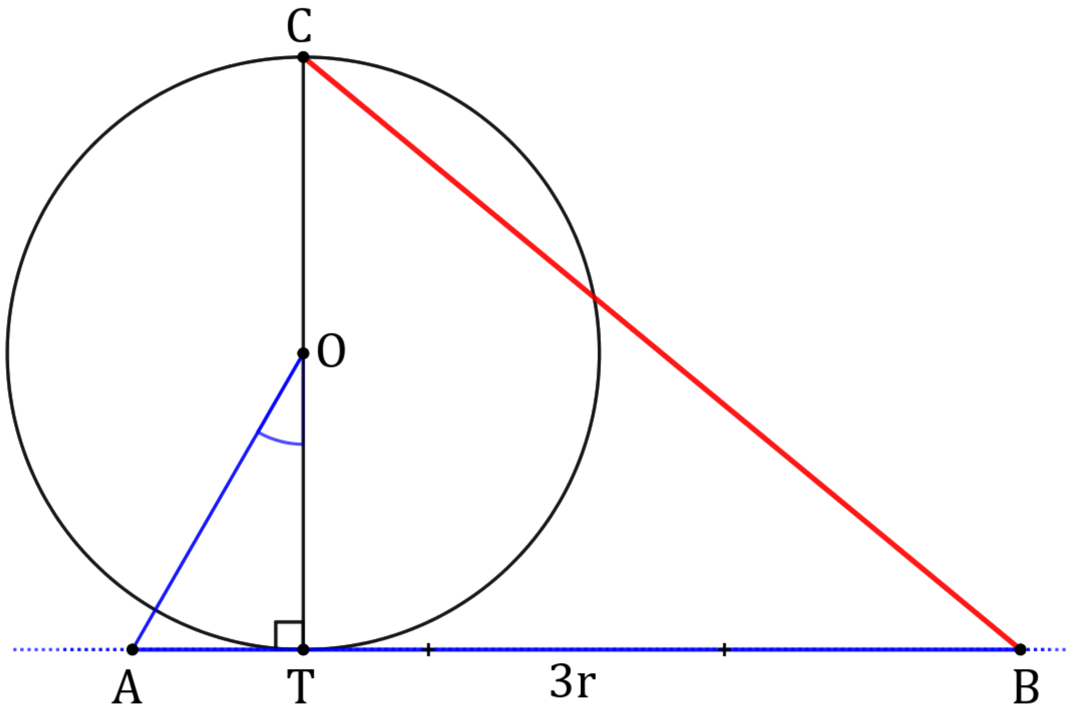
lecture

Kochansky's construction

A circle with center O and radius r is given. Approximate the half of its circumference ($\pi \cdot r$).

Steps of the construction

1. Draw a diameter: CT
2. Construct the tangent line at point T.
3. Construct an angle of 30° with vertex O and a ray OT.
4. The other ray intersects the tangent line at point A.
5. Copy the radius on ray AT three times.
   We get (end)point B.
6. The length of segment BC approximates
   the circumference of semicircle CT.



The sketch of the proof:

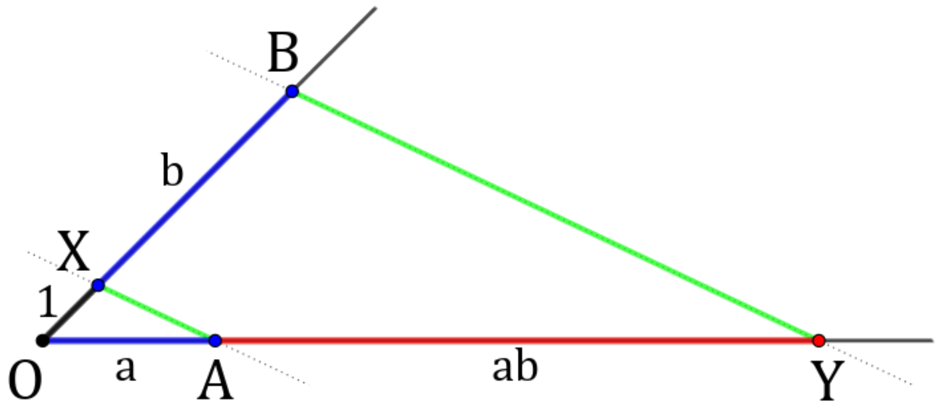
It is easy to see that $AT = \frac{r}{\sqrt{3}}$. (Use triangle $\triangle ATO$.) Then, $BC = \sqrt{(AB - AT)^2 + CT^2} = \cdots = \sqrt{\frac{40-6\sqrt{3}}{3}} \cdot r \approx 3.14153 \cdot r$

CONSTRUCTIONS OF ALGEBRAIC OPERATIONS – 1

Lengths a and b are given. Construct their product $a \cdot b$.

Steps of the construction

1. Draw an arbitrary angle with vertex O.
2. Copy length a on a ray: OA.
   Copy *1* (as length) and length b on the other ray: OX and XB.
3. Draw line AX.
   Draw parallel line to AX at point B.
4. This line intersects ray OA at point Y.
5. The length of AY is *ab*.



The main idea of the construction:

Using Intercept Theorem, we get $\frac{\mathrm{OA}}{\mathrm{AY}} = \frac{\mathrm{OX}}{\mathrm{XB}}$. Thus, $\mathrm{OA} \cdot \mathrm{XB} = \mathrm{OX} \cdot \mathrm{AY}$ $(a \cdot b = 1 \cdot \mathrm{AY})$.

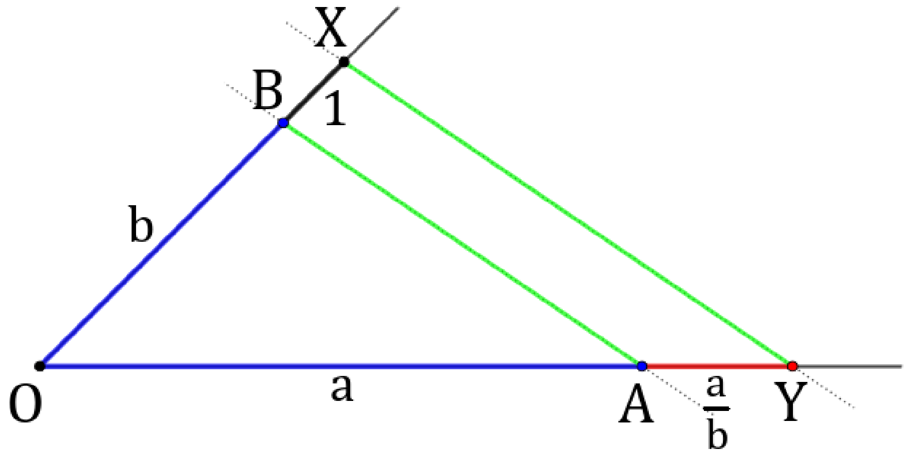
lecture

Constructions of algebraic operations – 2

Lengths a and b are given. Construct their quotient $\frac{a}{b}$.

Steps of the construction

1. Draw an arbitrary angle with vertex O.
2. Copy length a on a ray: OA.
   Copy lengths b and 1 on the other ray: OB and BX.
3. Draw line AB.
   Draw parallel line to AB at point X.
4. This line intersects ray OA at point Y.
5. The length of AY is $\frac{a}{b}$.



The main idea of the construction:

Using Intercept Theorem, we get $\frac{\mathrm{OA}}{\mathrm{AY}} = \frac{\mathrm{OB}}{\mathrm{BX}}$, i.e. $\frac{a}{\mathrm{AY}} = \frac{b}{1}$. Thus, $\frac{a}{b} = \mathrm{AY}$.

lecture

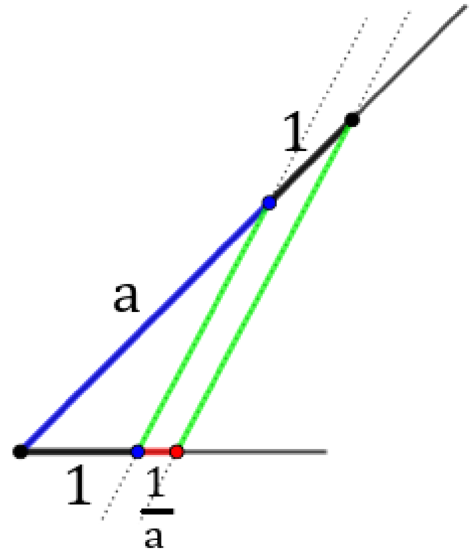
Constructions of algebraic operations – 3

Length a is given. Construct its reciprocal $\frac{1}{a}$.

Steps of the construction

Using the previous construction,

1. Draw an arbitrary angle with vertex O.
2. Copy *1* (as length) on a ray.
   Copy lengths a and *1* on the other ray.
3. Use Intercept Theorem to get $\frac{1}{a}$.



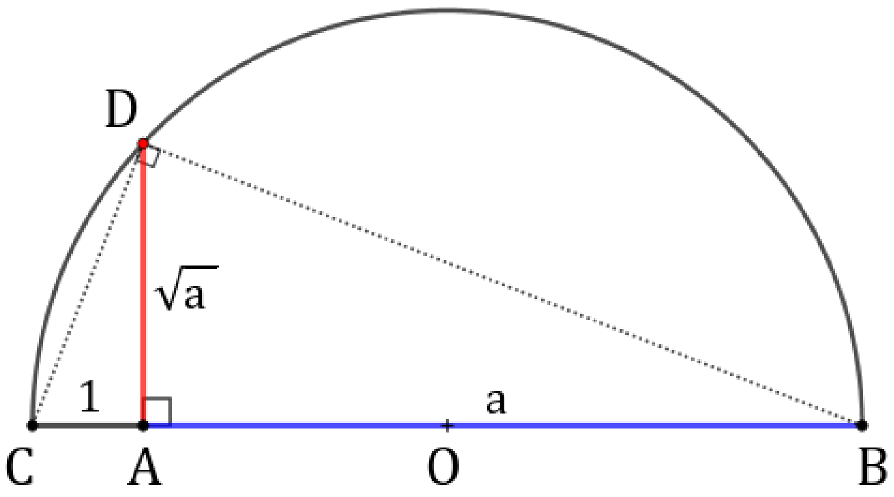
lecture

Constructions of algebraic operations – 4

Length a is given. Construct its square root $\sqrt{a}$.

Steps of the construction

1. Draw an arbitrary line, then copy *1* (as length) and length a on this line. (We get segments CA and AB.)
2. Draw a semicircle with diameter CB.
3. Draw a perpendicular line to CB which passes through A.
4. This line intersects the semicircle at point D.
5. The length of AD is $\sqrt{a}$.



The main idea of the construction:

Triangles $\triangle$CAD and $\triangle$DAB are similar to each other. Therefore, $\frac{AB}{AD} = \frac{AD}{AC}$.
It means that $AB \cdot AC = a \cdot 1 = AD^2$. Thus, $AD = \sqrt{a}$.

(See Geometrical Constructions 1 – Basic theorems about triangles)

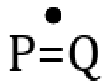
lecture

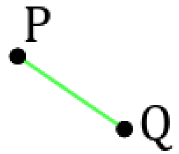
METRIC PROBLEMS IN SPACE – 1

Two points

- P and Q coincide with each other, or
- P and Q are different.

The distance of P and Q is the length of segment PQ.

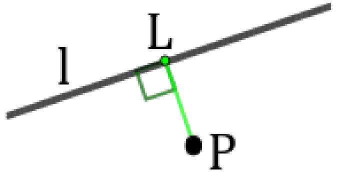


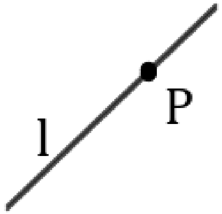


A point and a line

- P does not lie on line l, or
- P lies on line l.

The distance of P and l is the length of segment LP.
(L is the footpoint of the line which is perpendicular to l and passes through P.)



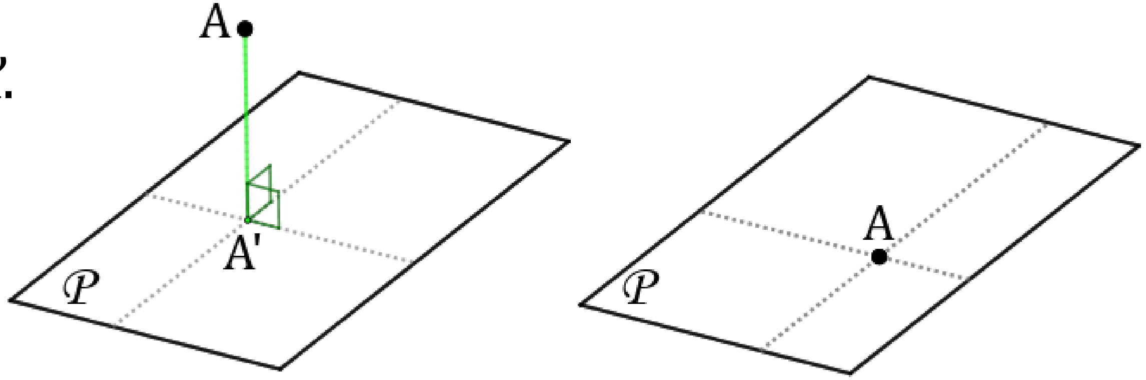


Metric problems in space – 2

A point and a plane

- Point A is not in plane $\mathcal{P}$, or
- point A is in plane $\mathcal{P}$.

The distance of A and $\mathcal{P}$ is the length of segment AA'.



A' is *the orthogonal projection* of A on $\mathcal{P}$.

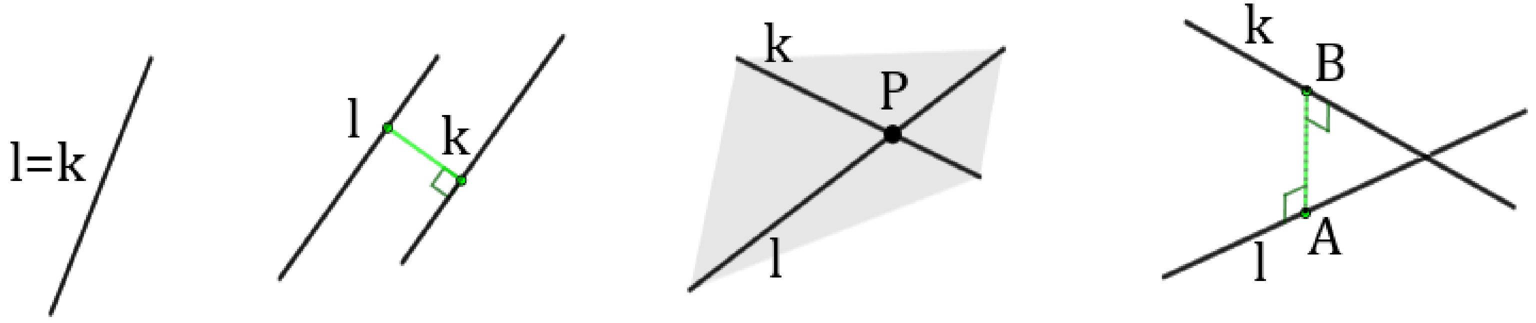
How to obtain A' ?

1. Draw a perpendicular line to plane $\mathcal{P}$ through A.
2. This line intersects the plane at point A'.

Metric problems in space – 3

Two lines can be

- the same (l=k),
- parallel,
- intersecting, or
- skew.



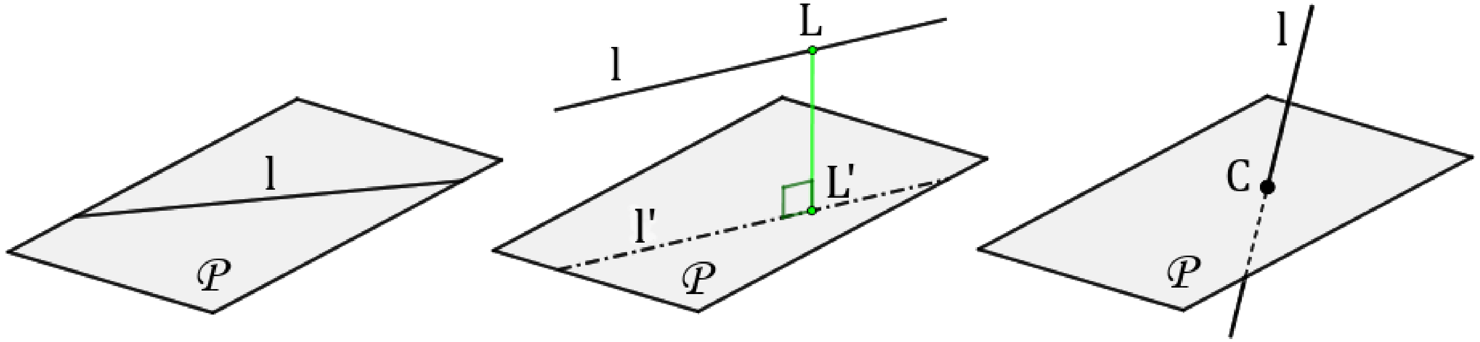
If l=k or these two lines are intersecting, then their distance is zero.

If l and k are parallel lines, then their distance is the distance of e.g. line l and an arbitrary point of line k.

If l and k are skew, then we can get their distance using a special line which is perpendicular to both lines. This line is the *normal transversal* of two lines. The normal transversal intersects l and k at points A and B, respectively. The distance of the lines is the length of segment AB.

Metric problems in space – 4

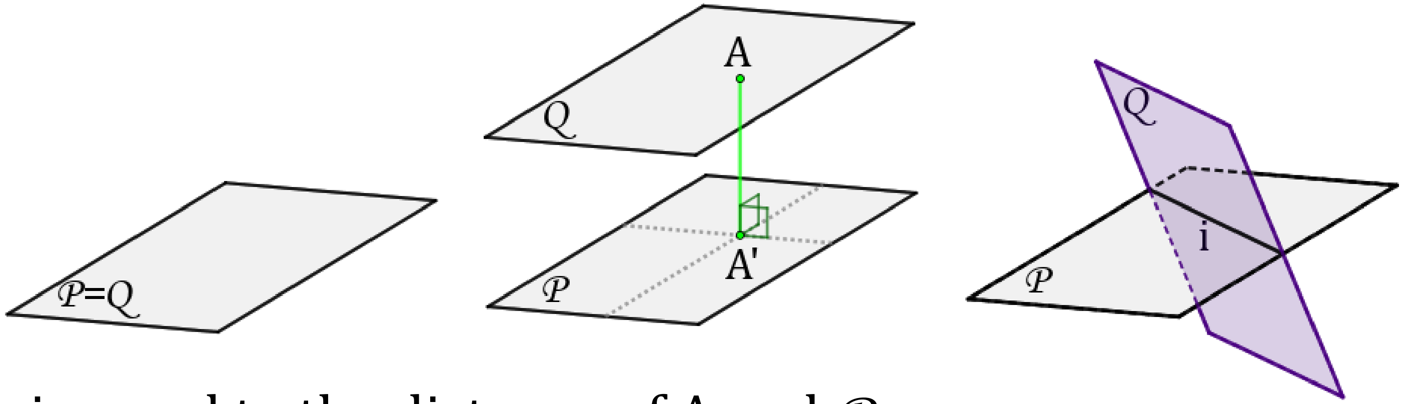
A line and a plane



- Line l is in plane $\mathcal{P}$.
- Line l is parallel to plane $\mathcal{P}$
  (if and only if there is a line in $\mathcal{P}$ so that this line is parallel to line l.)
- Line l intersects plane $\mathcal{P}$.

If l is parallel to $\mathcal{P}$, then their distance is equal to the distance between L and $\mathcal{P}$ (where L is an arbitrary point on l).

Two planes can be



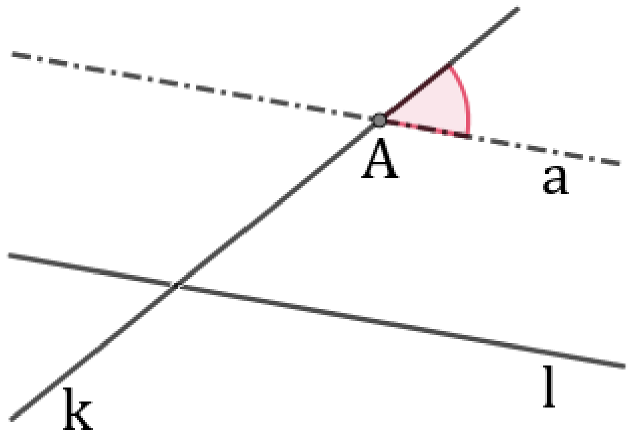
- the same,
- parallel, or
- intersecting.

If $\mathcal{P}$ an Q are parallel, then their distance is equal to the distance of A and $\mathcal{P}$ (where A is an arbitrary point in Q).

Metric problems in space – 5

The angle of two lines

Cases of parallel or intersecting lines are obvious.



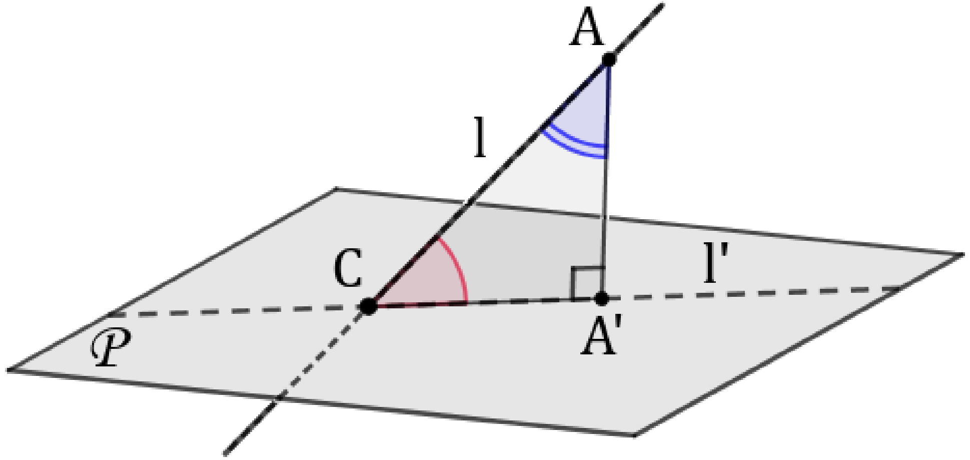
If k and l are skew, then we can use an arbitrary line which is parallel to e.g. line l and intersects line k.

The measure of the angle of k and l is equal to the measure of the angle of k and a.

Important remark: The measure of the angle of two lines is less or equal than 90°.

Metric problems in space – 6

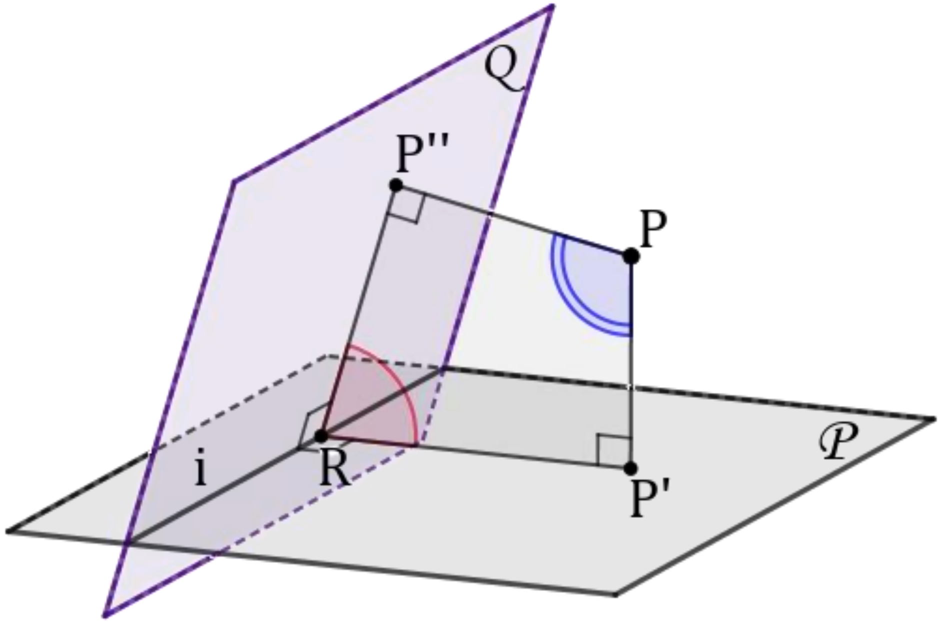
The angle of a line and a plane is the angle of the line and its orthogonal projection on the plane.



1. Let A be an arbitrary point on line l.
2. A’ is the orthogonal projection of A on $\mathcal{P}$.
3. Calculate or construct the angle of lines l and AA’ instead of the angle of line l and plane $\mathcal{P}$.
4. Triangle AA’C is a right triangle, therefore ∢ACA’+∢CAA’=90°.
5. It means that
   ∢(l, $\mathcal{P}$) = ∢ACA’ = 90° – ∢CAA’

Metric problems in space – 7

The angle of two planes ($\mathcal{P}$, Q) is the angle of two lines which are perpendicular to the intersection line of the planes (i) and lie on $\mathcal{P}$ and Q, respectively.

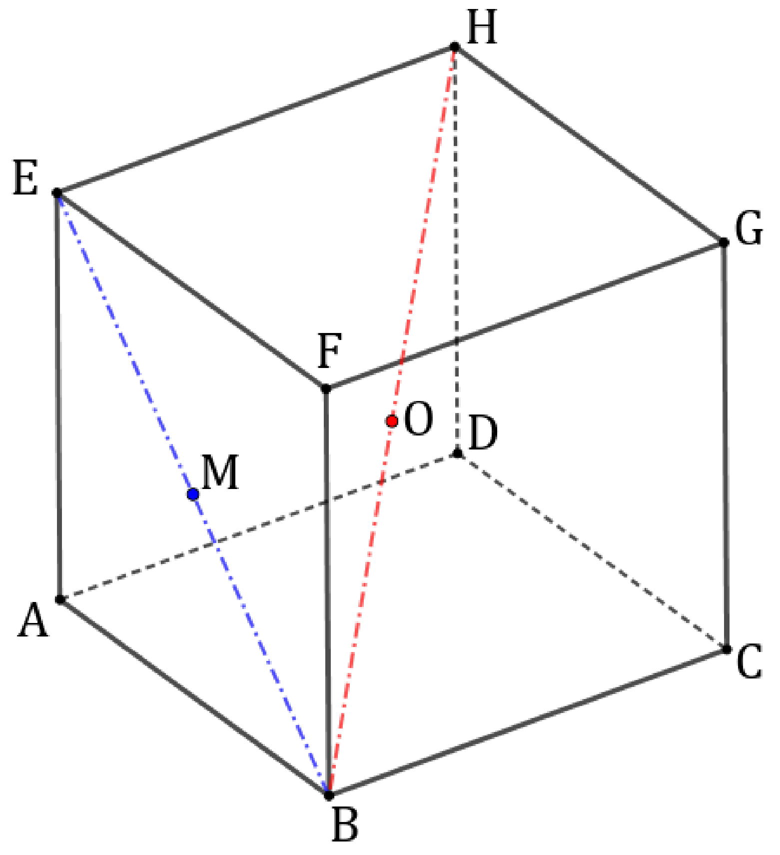


1. Let P be an arbitrary point which does not lie on plane $\mathcal{P}$ or plane Q.
2. P’ and P’’ are the orthogonal projections of A on $\mathcal{P}$ and Q, respectively.
3. R is the intersection point of line i and plane [P,P’,P’’].
   (Line i is perpendicular to plane [P,P’,P’’].)
4. Calculate or construct the angle of lines PP’ and PP’’ instead of the angle of the two planes.
5. ∢PP’R and ∢ PP’’R are right angles, so
   ∢ P’PP’’+ ∢ P’RP’’=90° (PP’RP’’ is a cyclic quadrilateral.)
6. It means that ∢P’RP’’ = 180° – ∢P’PP’’ . Therefore, ∢($\mathcal{P}$, Q) = ∢(P’R,P’’R) = ∢(PP’,PP’’) .

POLYHEDRA

A **polyhedron** is a solid shape with four or more flat surfaces.

Example: a cube (see the figure)



Vertices: e.g. A, B, C, …
Edges: e.g. AB, CG, …
Faces: e.g. ABCD, ABFE, …

Euler's polyhedron formula:
Consider a polyhedron. If v is the number of vertices, e is the number of edges, and f is the number of faces then

$$v - e + f = 2$$

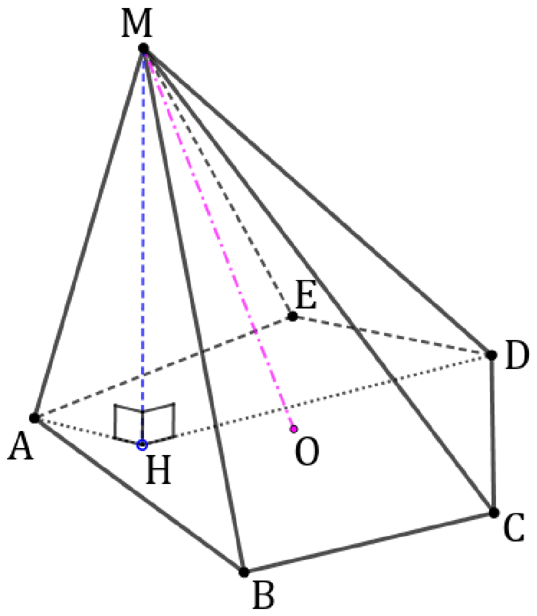
Further terminology:
Face diagonal: e.g. BE
Space (or body) diagonal: e.g. BH
Center/centre: O

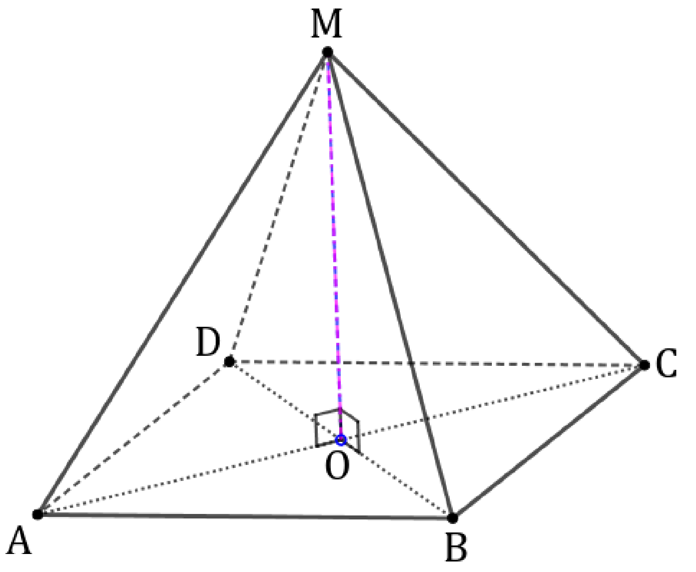
Pyramids

A **pyramid** is a polyhedron formed by lines (or segments) connecting the points of a polygon and a point which does not lie on the plane of the given polygon.

Base: e.g. ABCDE
Apex: point M
Axis: line OM

Lateral face: e.g. triangle ABM
Lateral edge: e.g. AM





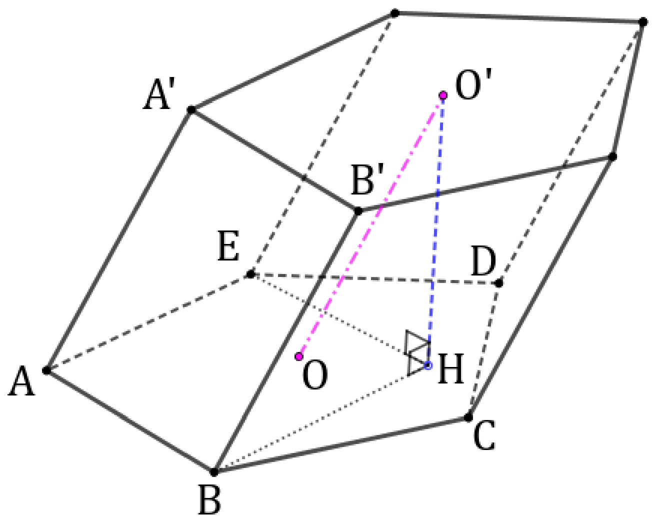
Right pyramid: The orthogonal projection of the apex on the plane of the base is the centroid of the base. A non-right pyramid is called an *oblique pyramid*.
(Regular pyramid: The base of the right pyramid is a regular polygon.)

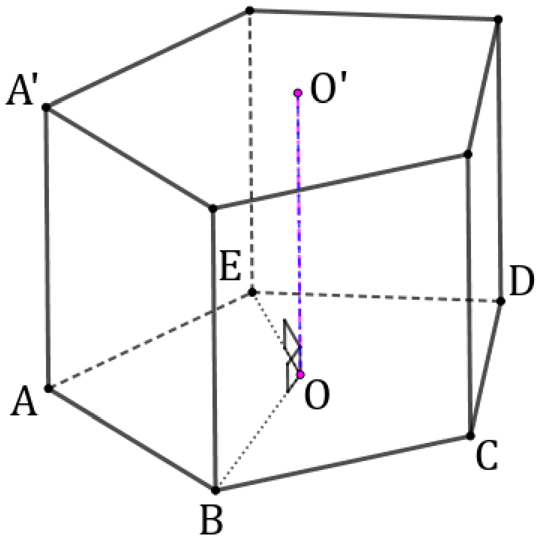
Prisms

Consider a polygon and a line (or a segment) which does not lie on the plane of the polygon.
A **prism** is a polyhedron formed by the lines (or segments) which are parallel to the given line (or segment) and pass through the points of the given polygon.

Lower base: e.g. ABCDE
Upper base: e.g. A'B'C'D'E'
Axis: e.g. OO'
(O and O' are the centroids of the bases.)

Lateral face: e.g. parallelogram ABB'A'
Lateral edge: e.g. AA'





The lateral edges of a *right prism* are perpendicular to the plane of its base. A non-right prism is called an *oblique prism*.
(Regular prism: The base of the right prism is a regular polygon.)

REGULAR POLYHEDRA – 1

A **regular polyhedron** (or Platonic solid) has congruent, regular polygonal faces so that all of its edges have the same length and all of its angles of faces have the same measure.

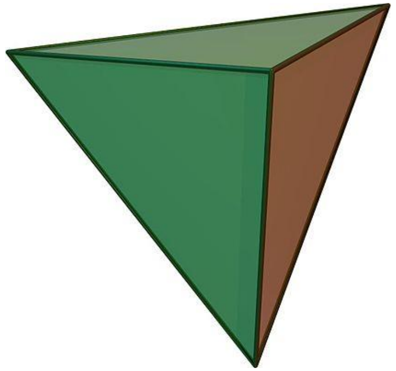
Using Euler's polyhedron formula, it can be proved that *five regular polyhedra exist*:

Regular polyhedron	*Faces*		*Vertices*	*Edges*	*Dual polyhedron*
Tetrahedron	equilateral triangle	4	4	6	Tetrahedron
Cube (hexahedron)	square	6	8	12	Octahedron
Octahedron	equilateral triangle	8	6	12	Cube
Dodecahedron	regular pentagon	12	20	30	Icosahedron
Icosahedron	equilateral triangle	20	12	30	Dodecahedron

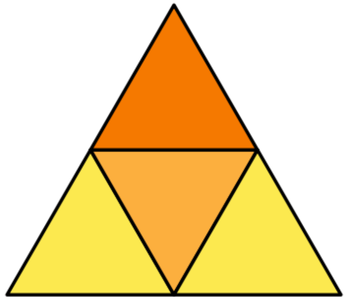
Every regular polyhedron has a *dual polyhedron* with faces and vertices interchanged.

Regular polyhedra – 2

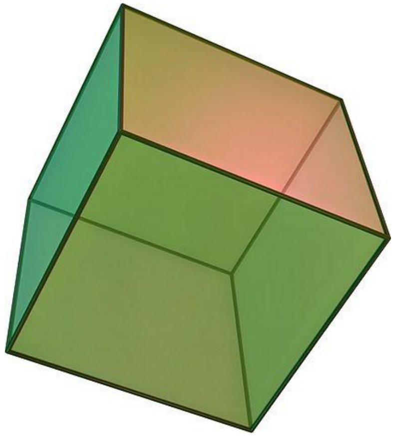
Tetrahedron



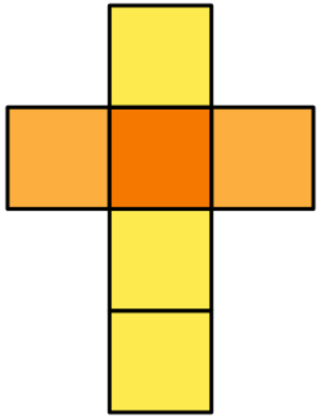
net



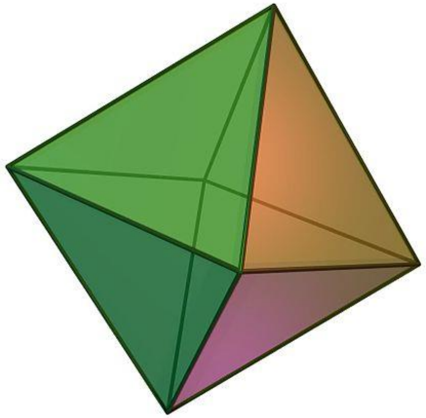
Cube



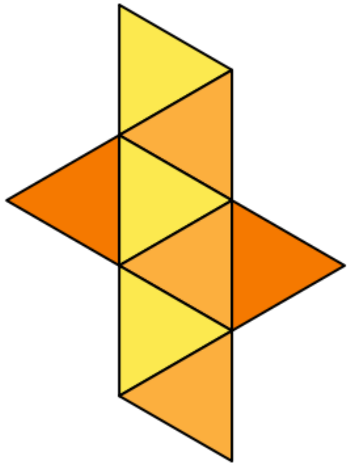
net



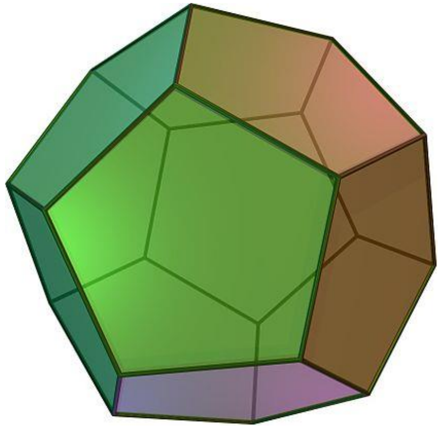
Octahedron



net



Dodecahedron



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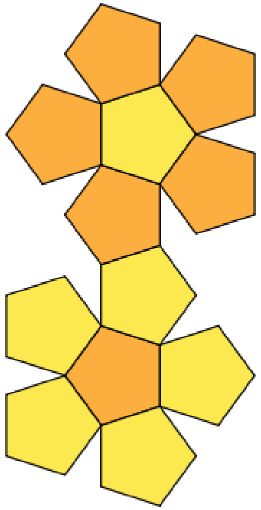
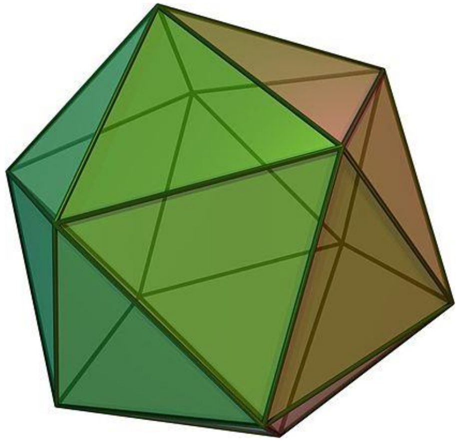
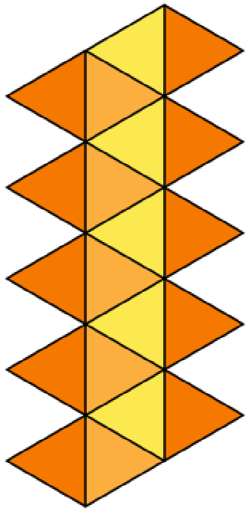


Icosahedron



net

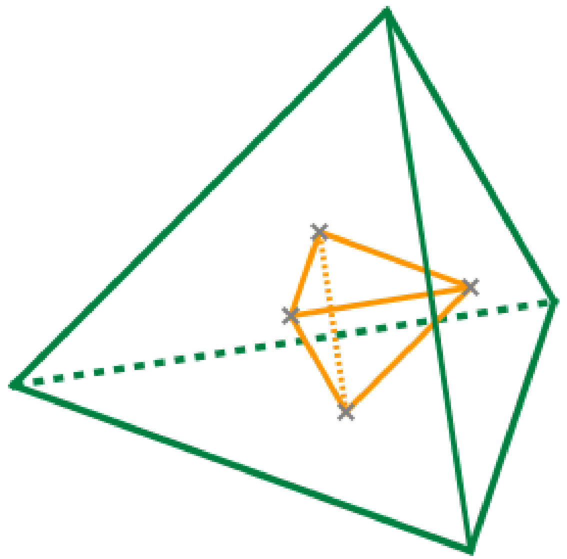


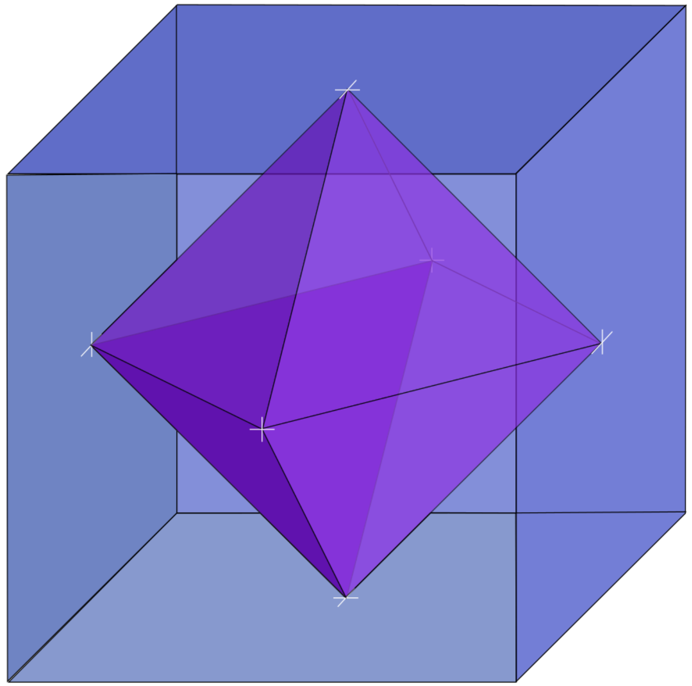
sources: Wikipedia

Regular polyhedra – 3

Examples of duality

The centroids of the faces of a regular polyhedron form its dual polyhedron.





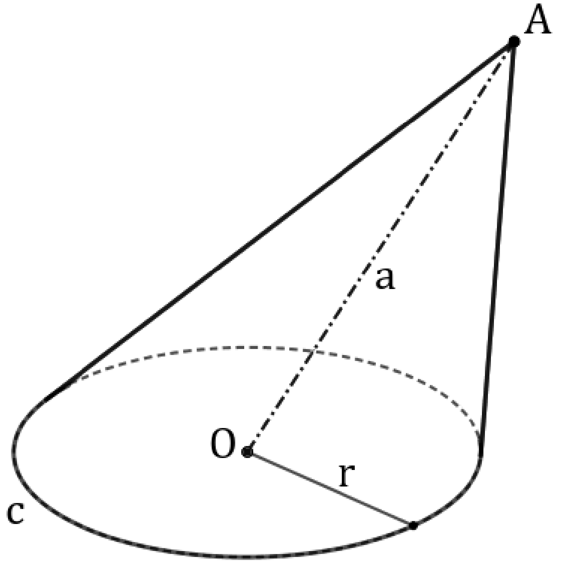
sources: Wikipedia

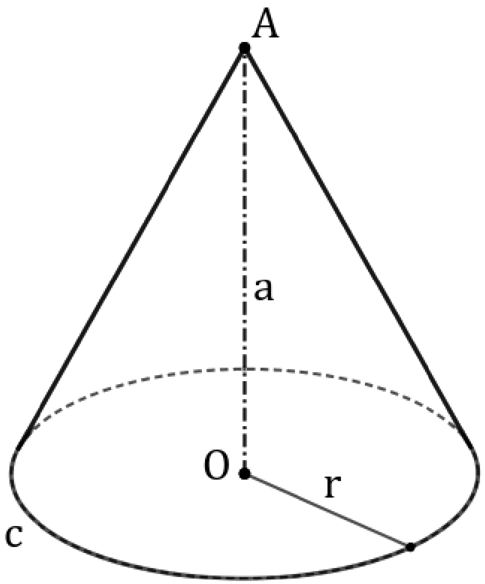
CONE, CYLINDER, SPHERE – 1

A (circular) **cone** is a three-dimensional shape formed by lines (or rays or segments) connecting the points of a circle and a point which does not lie on the plane of the given circle.

Base: circle c (with center O and radius r)
Apex: point A
Axis: line/segment a

Generatrix (generator line):
connecting lines/rays/segments
Lateral surface: the set of generatrixes





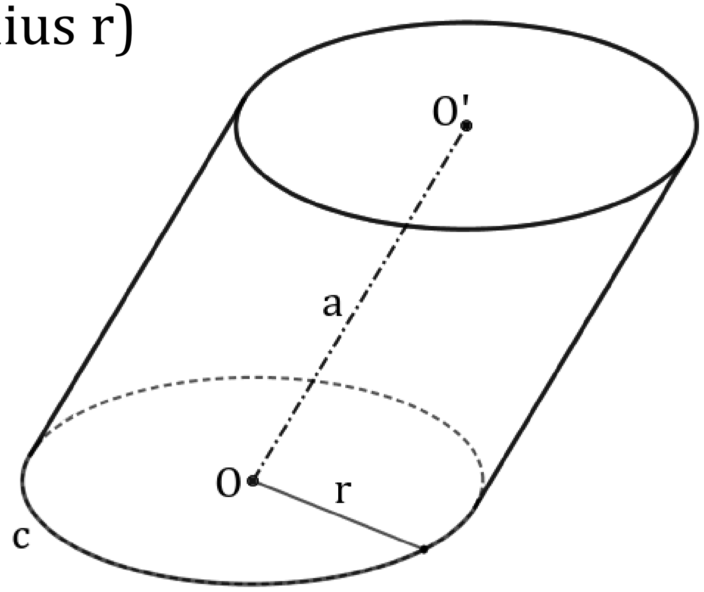
Right cone: The orthogonal projection of apex A on the base plane is center O.
A non-right cone is called an *oblique cone*.

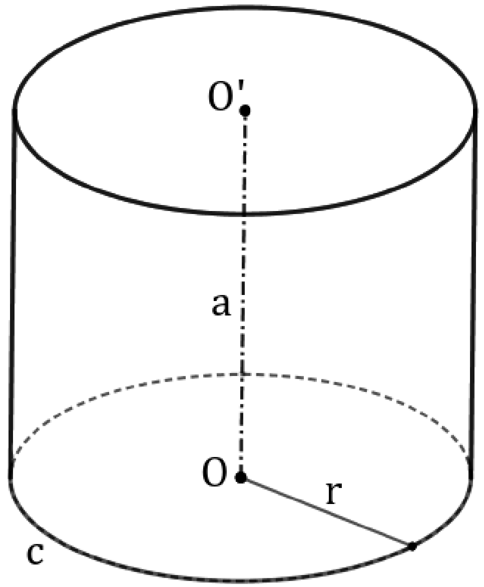
Cylinder, cone, sphere – 2

Consider a circle and a line (or a segment) which does not lie on the plane of the circle.
A (circular) **cylinder** is a three-dimensional shape formed by the lines (or segments) which are parallel to the given line (or segment) and pass through the points of the given circle.

Lower base: circle c (with center O and radius r)
Upper base: circle with center O'
Axis: OO'

Generatrix (generator line):
connecting lines/rays/segments
Lateral surface: the set of generatrixes

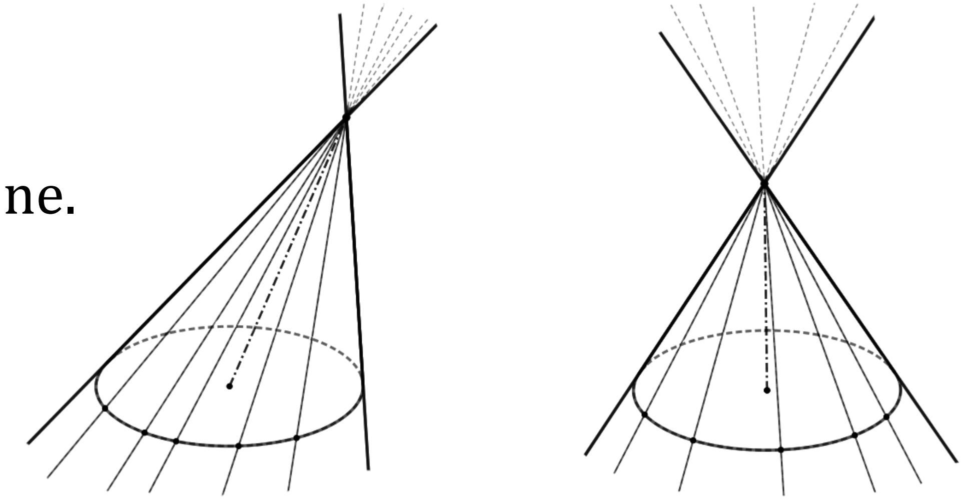




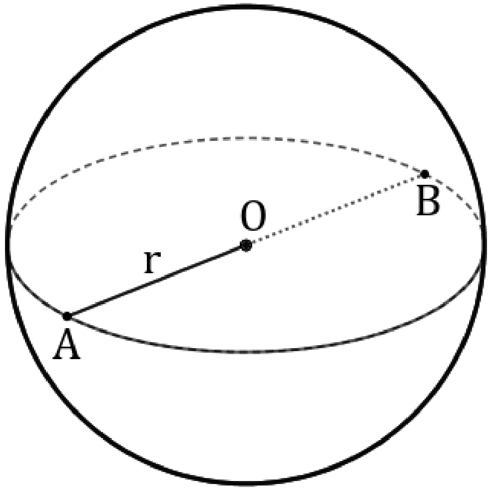
Right cylinder: The orthogonal projection of center O' on the lower base plane is center O.
A non-right cylinder is called an *oblique cylinder*.

Cylinder, cone, sphere – 3

Remark:
If a cone is generated by lines, then we obtain a double cone.
Its upper and lower parts are called *nappes*.



A **sphere** is the set of points in the three-dimensional space which are all at the same distance r from a given point O.



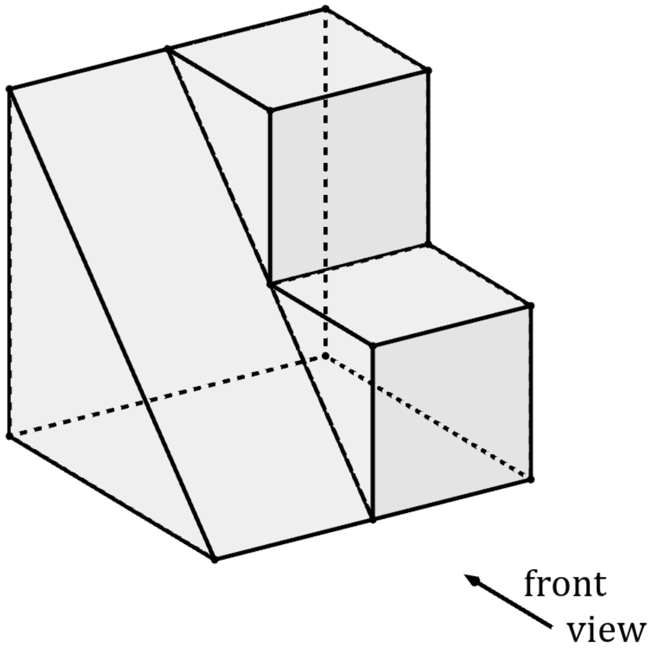
Center: O
Radius: r

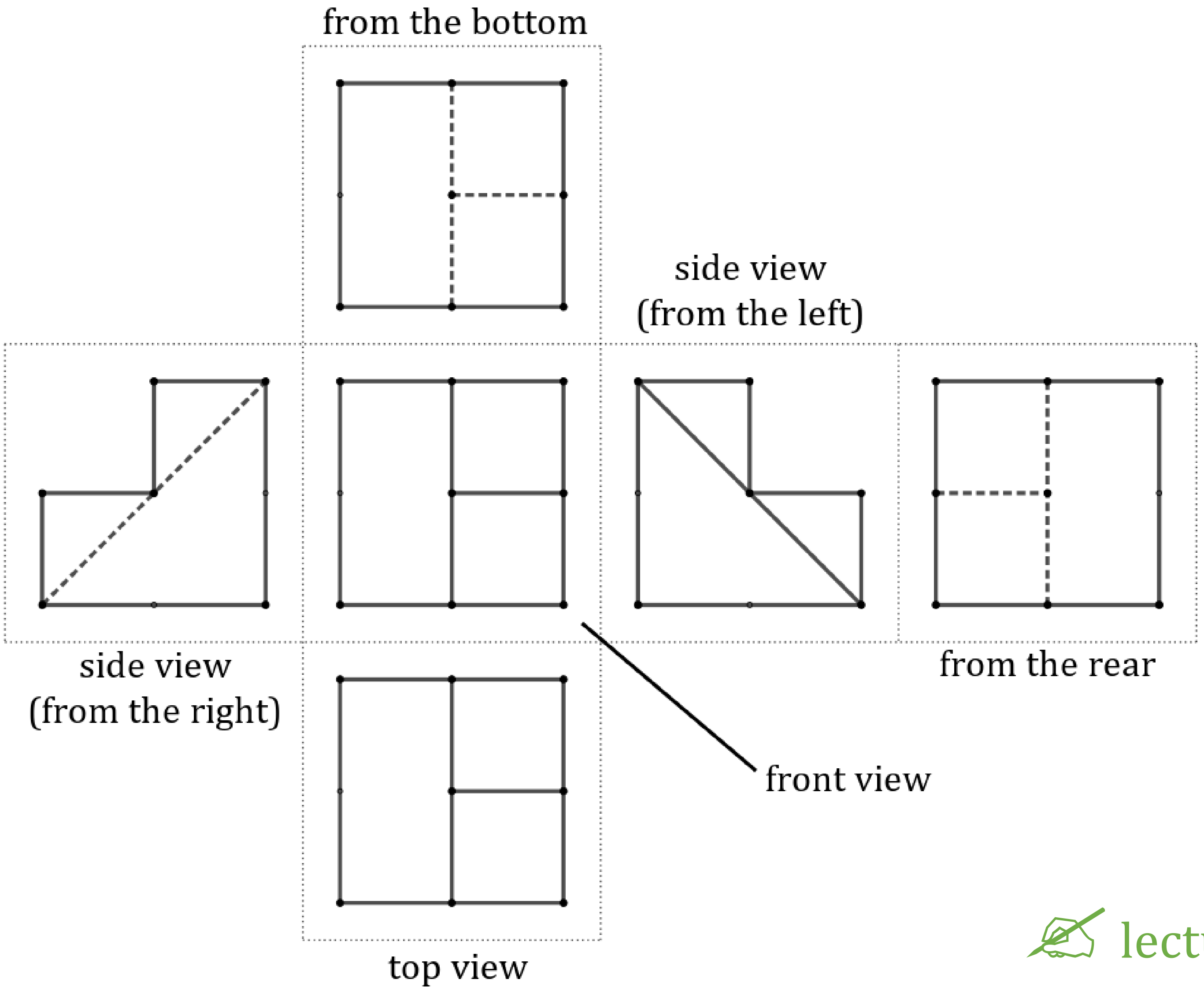
Great circle: a circle on the sphere with center O and radius r
Antipodal points: e.g. A and B

Representation of 3D – Motivation 1

Multi-view representation

Example 1



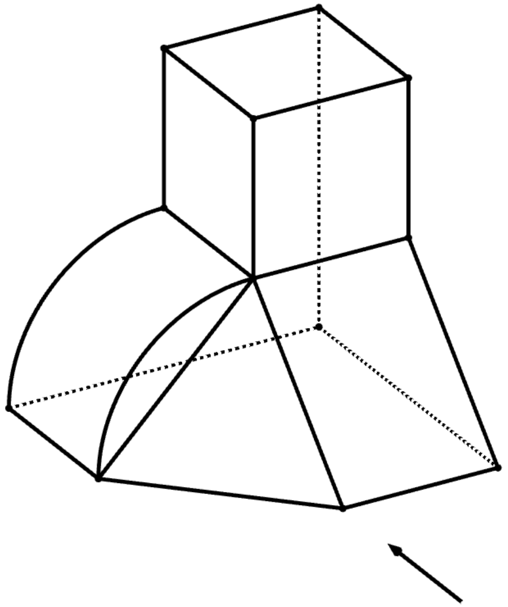


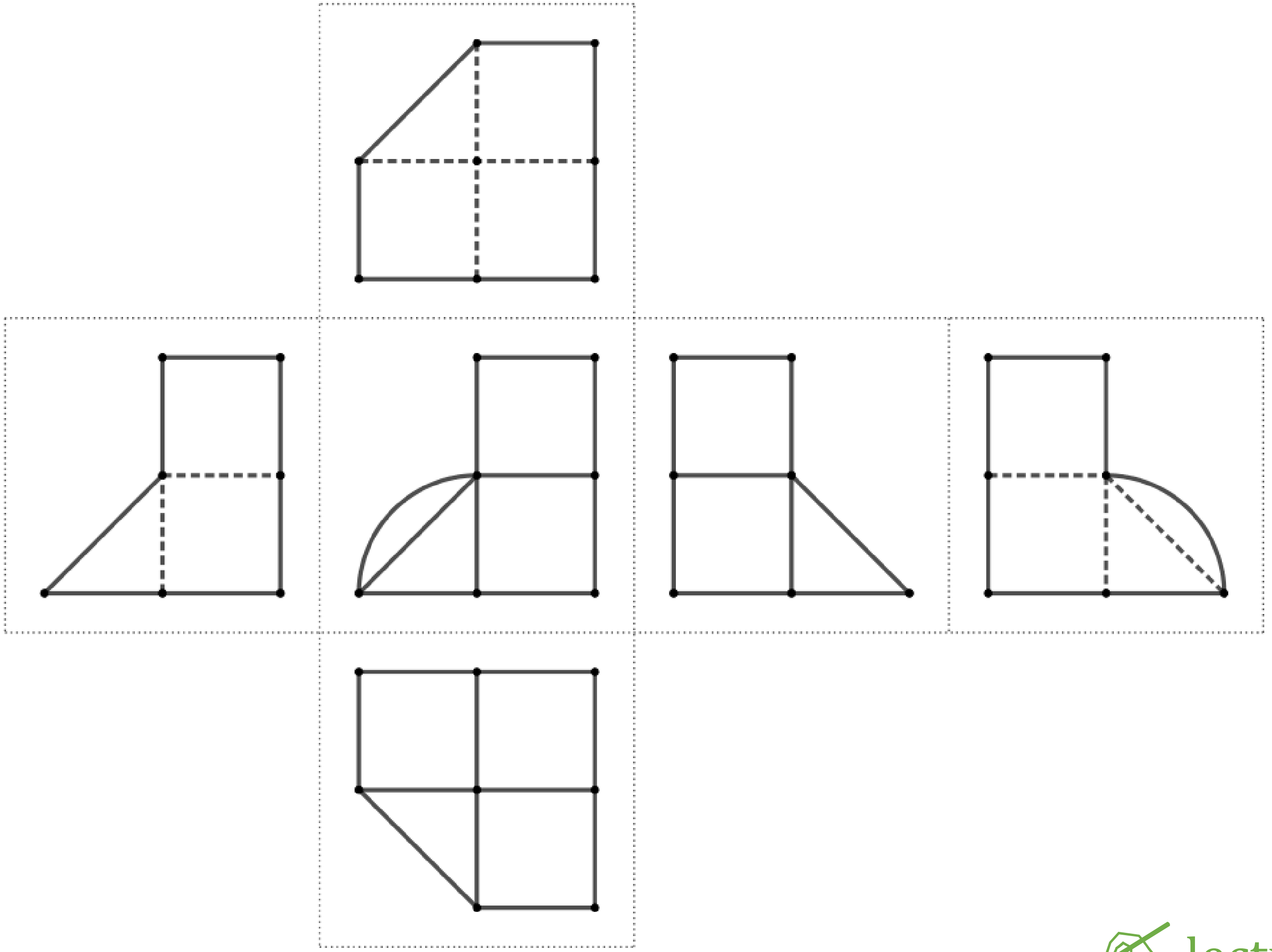
lecture

Representation of 3D – Motivation 2

Multi-view representation

Example 2



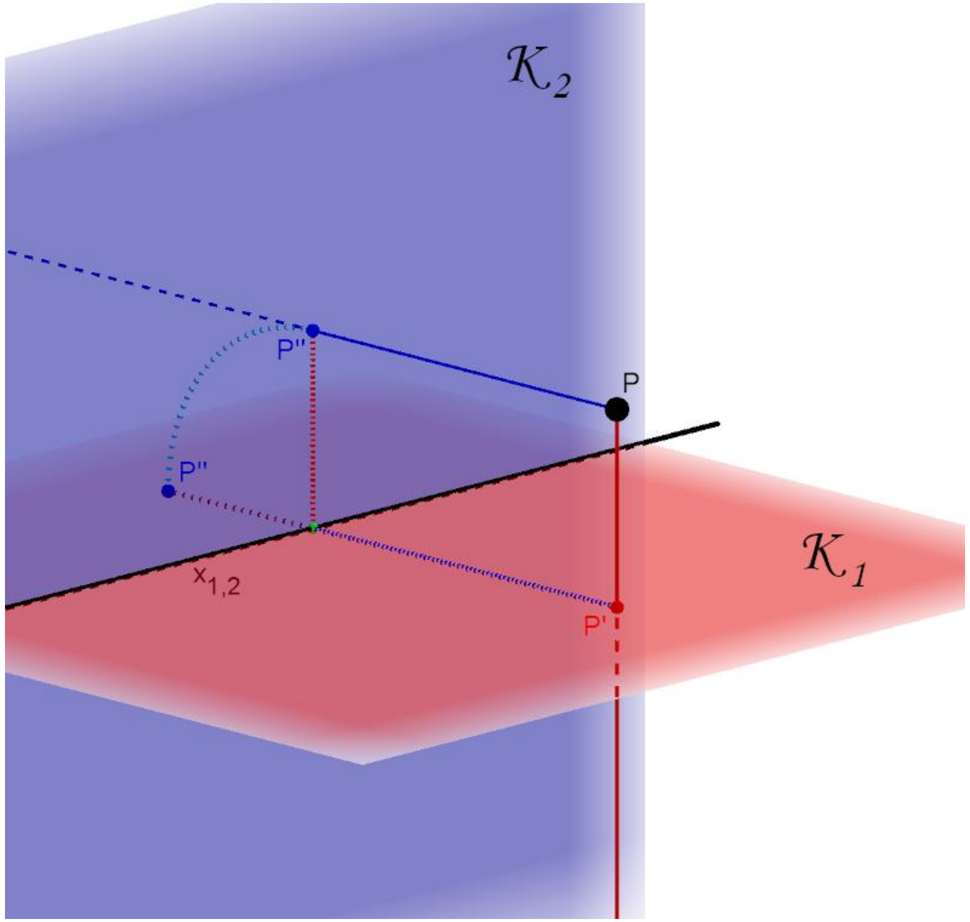


lecture

MULTI-VIEW REPRESENTATION

Consider two planes $\mathcal{K}_1$ and $\mathcal{K}_2$ which are perpendicular to each other.
Their intersection is line $x_{1,2}$.

We can represent an arbitrary point P using its *orthogonal projections* P' and P'' on $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively.
(orthogonal projection – see page 51)



Terminology:

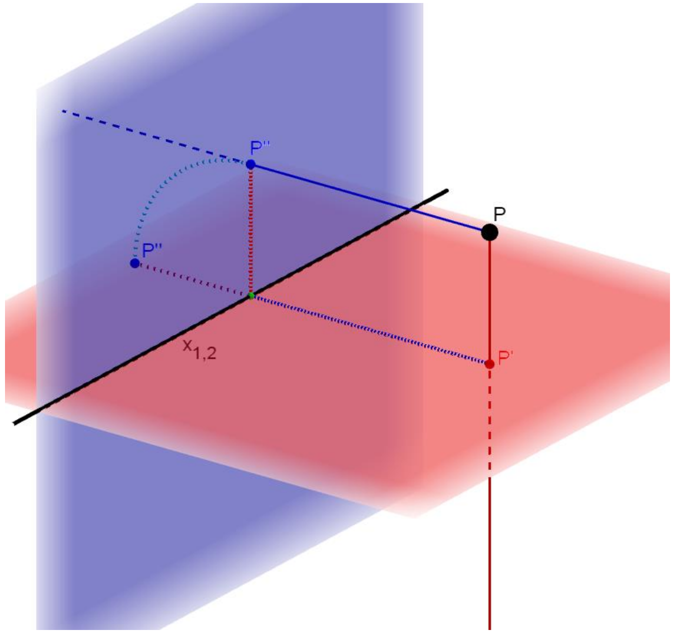
$\mathcal{K}_1$ – first image plane
$\mathcal{K}_2$ – second image plane
$x_{1,2}$ – (horizonal) axis
P' – first image of P / top view of P
P'' – second image of P / front view of P

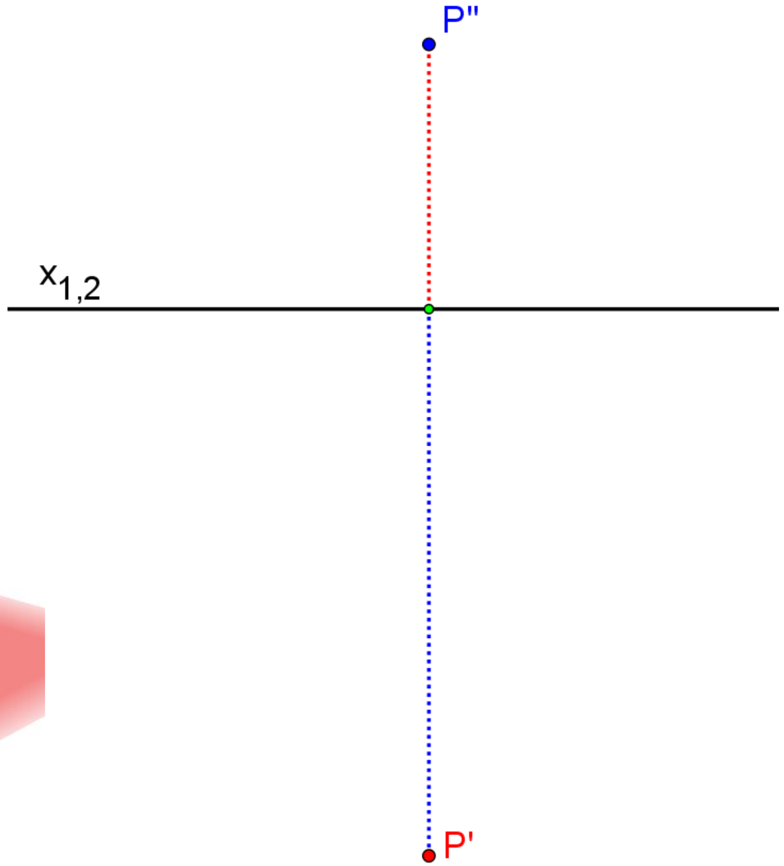
[Downloadable gif file](#) / [Interactive version](#)

Images of a point

We can represent both image planes in a single plane with the following method:

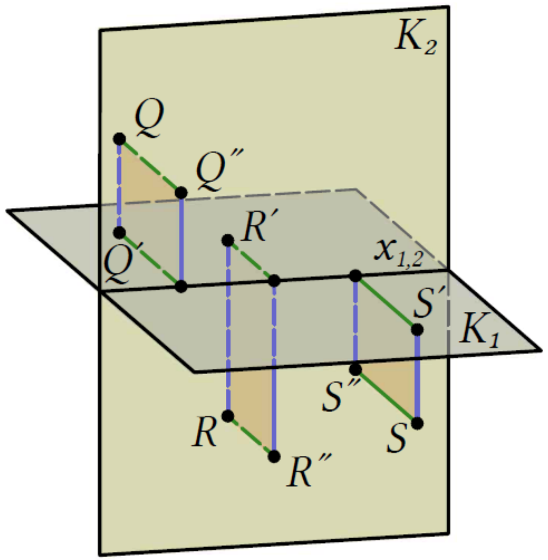
1. Construct both images of point P.
   The first image is in the first image plane, and
   the second image is in the second image plane.
2. *Fold the second image plane on the first image plane:*
   rotate the second image plane by 90° around the axis.
   (See the smaller figure.)
3. Points P, P', and P'' form
   a rectangle. The plane of
   this rectangle is perpendicular
   to the axis $x_{1,2}$.
4. Thus, after the folding,
   line P'P'' is also perpendicular
   to the axis $x_{1,2}$.
5. Now we can see both images
   in one plane.

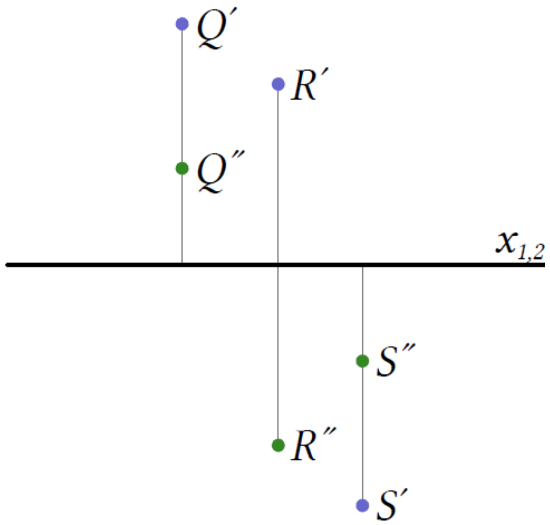




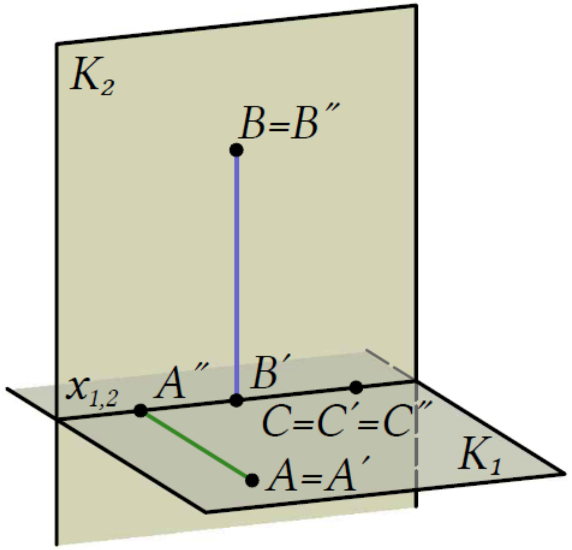
Images of some points

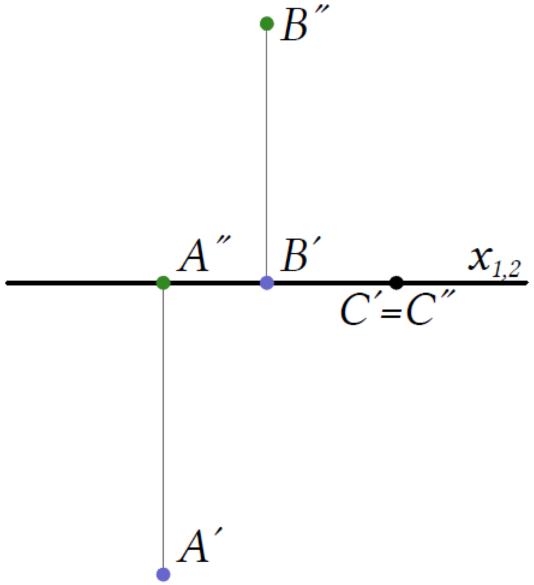
Points under $\mathcal{K}_1$ or behind $\mathcal{K}_2$:





Points in the image planes:





Source: Pék-Strommer – Ábrázoló geometria
(in Hungarian, for high school students)

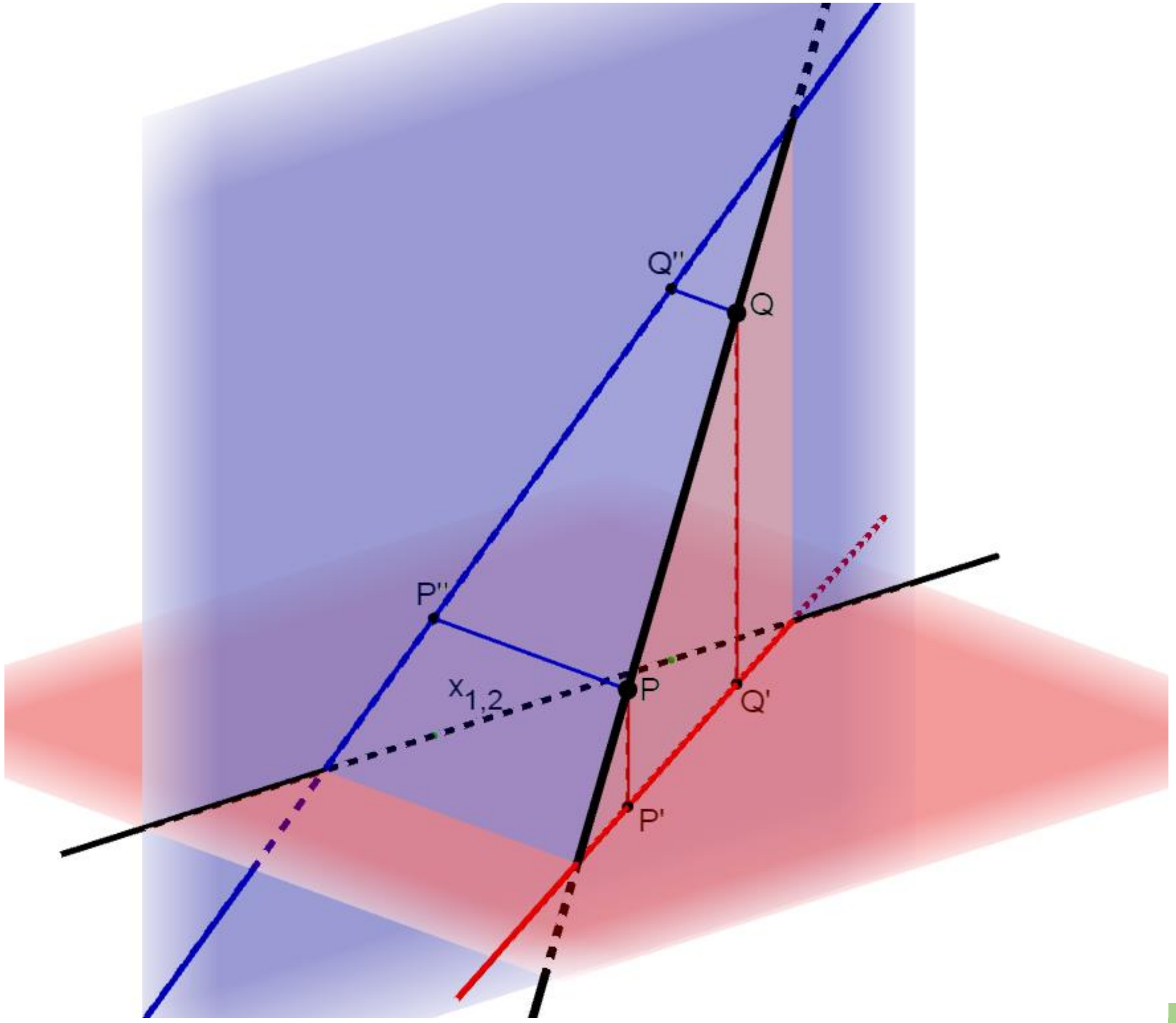
Images of a line – 1

A line can be represented
by its two arbitrary points.

The images of a line are two lines
in first and second image planes,
respectively.

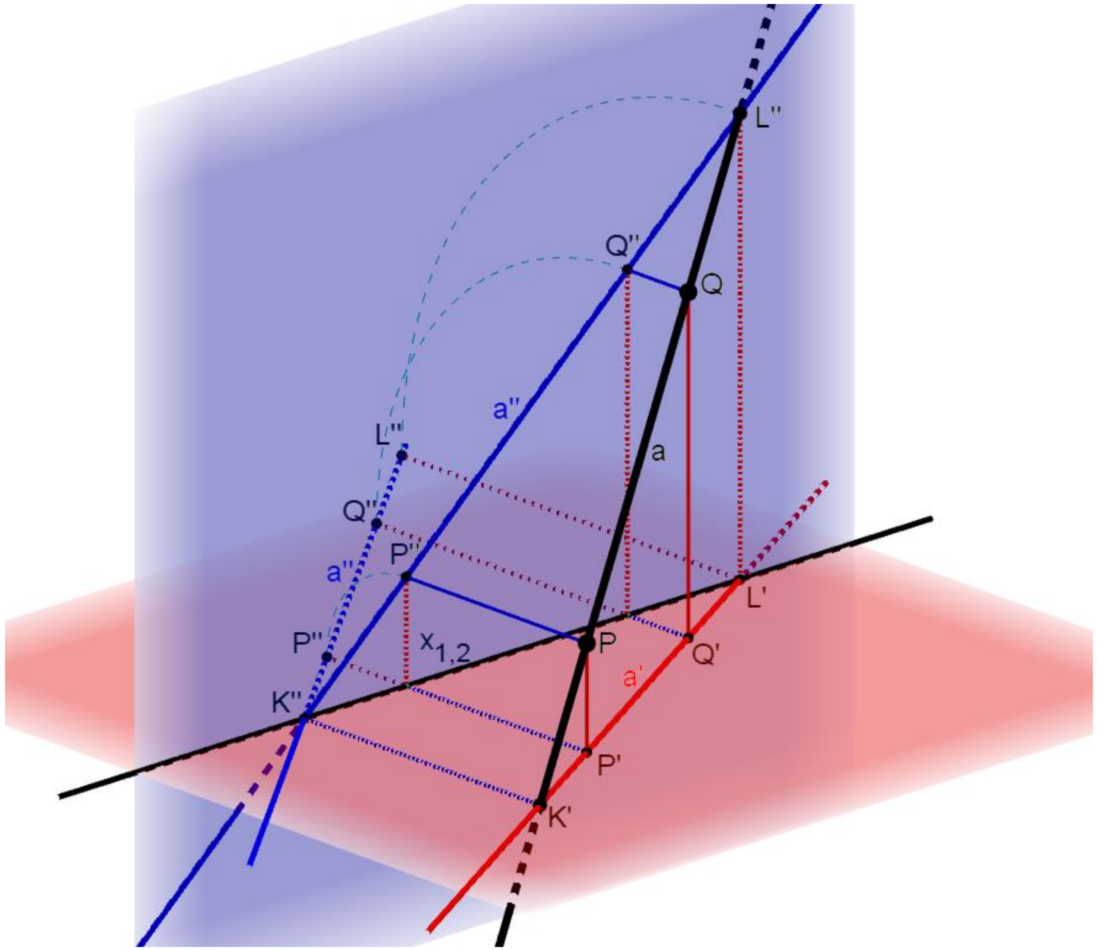
[Downloadable gif file]() / [Interactive version]()

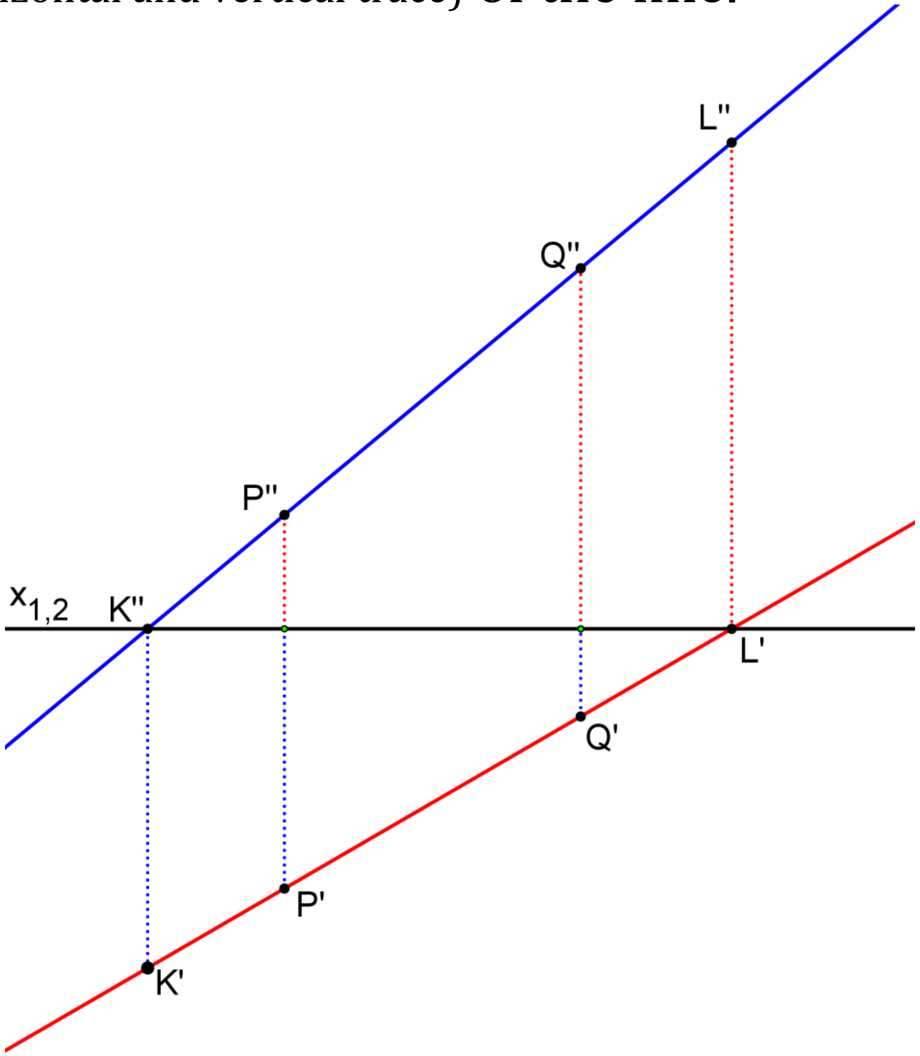
lecture



Images of a line – 2

Remark: Intersection points (K, L) of line a and the image planes have interesting images. These points are *the first and second tracing points* (or horizontal and vertical trace) of the line.



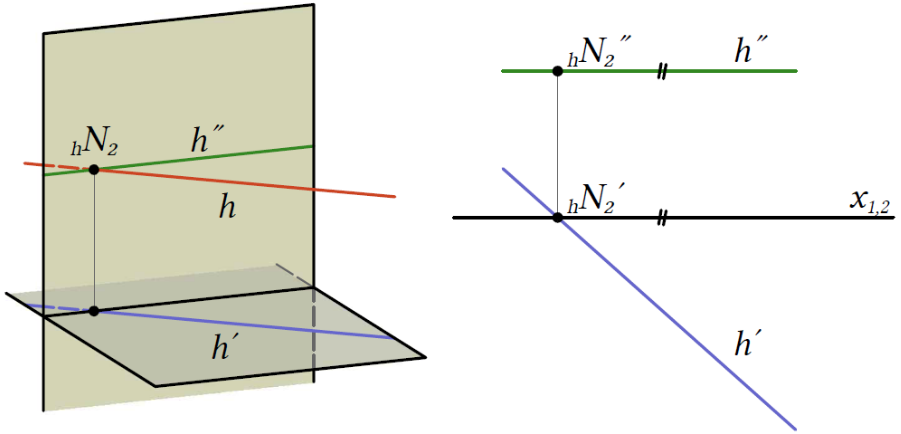


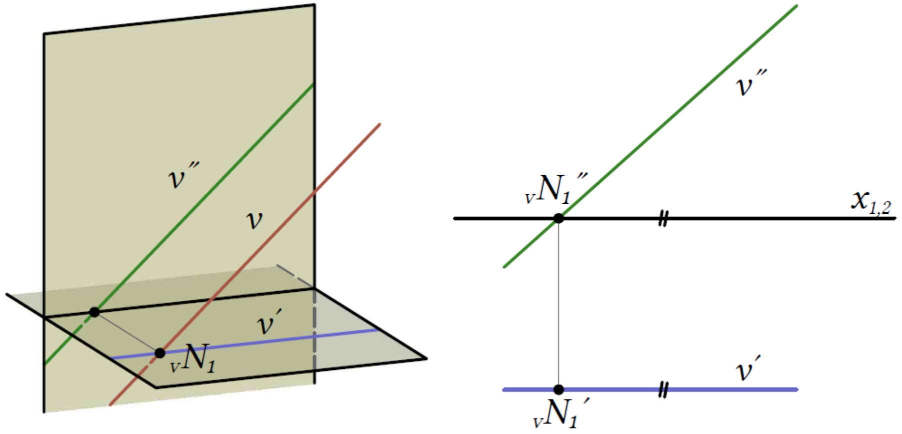
Special lines – 1

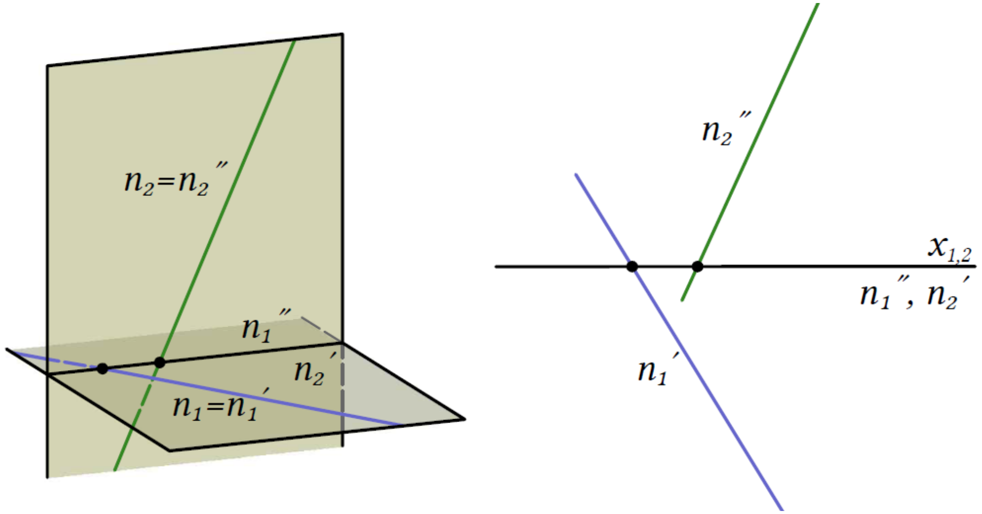
Lines which are parallel to image planes

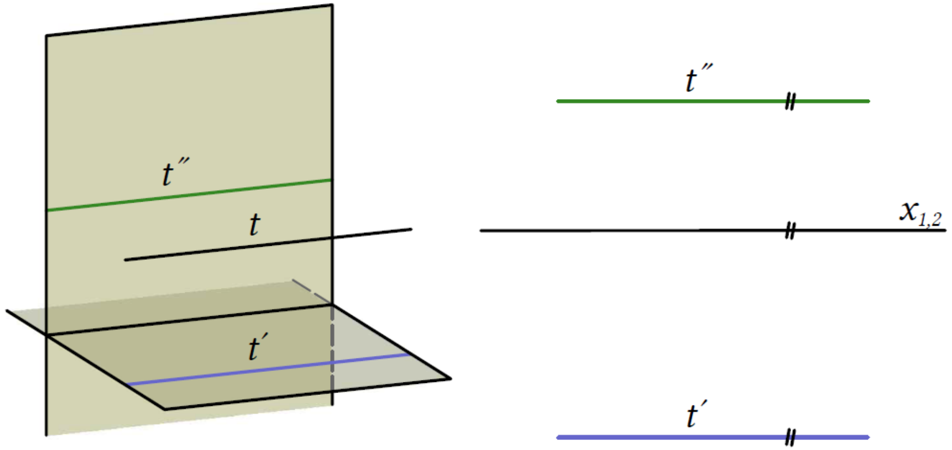
First/second principal lines – line h / line v

(or horizontal/vertical line or level/frontal line)







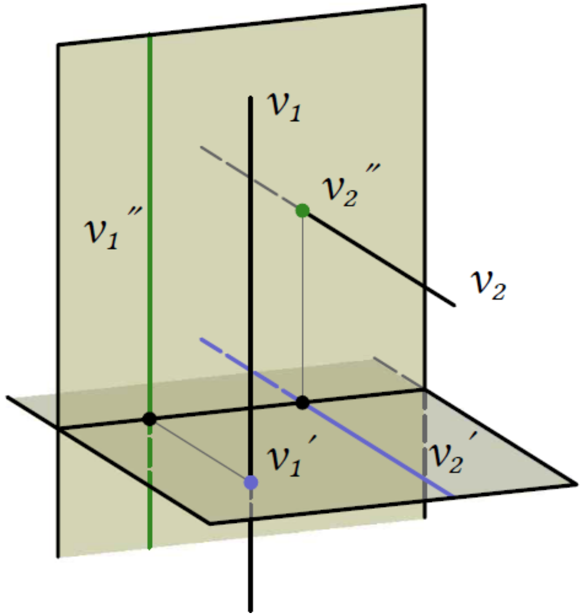


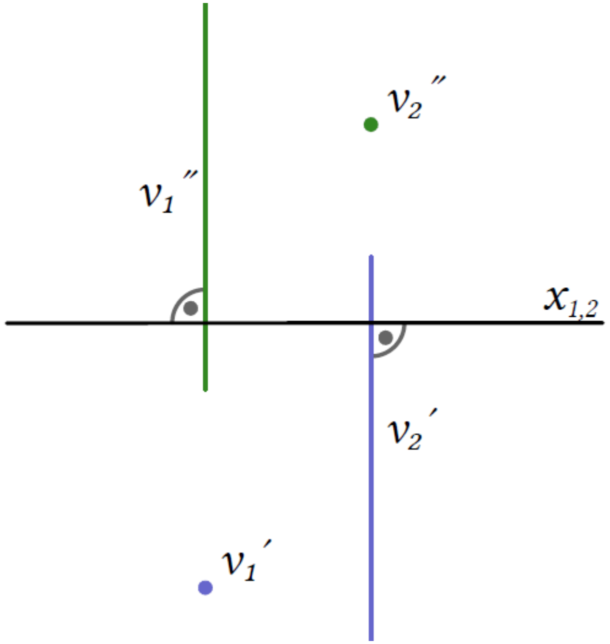
Source: Pék-Strommer – Ábrázoló geometria
(in Hungarian, for high school students)

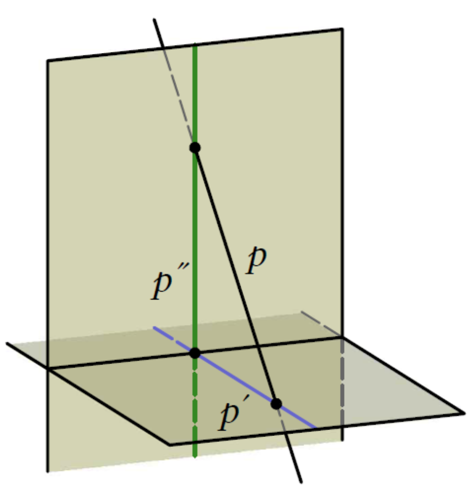
Special lines – 2

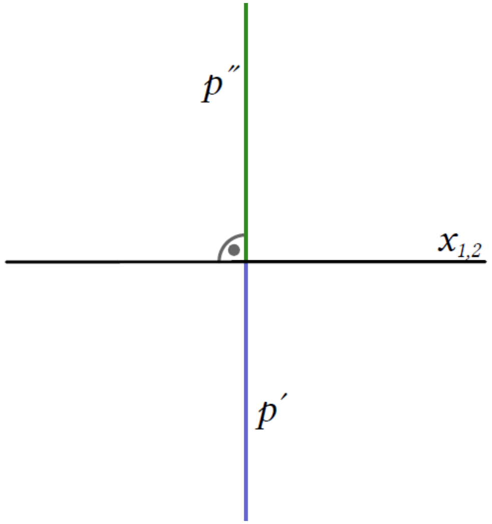
Lines which are perpendicular to image planes or the axis

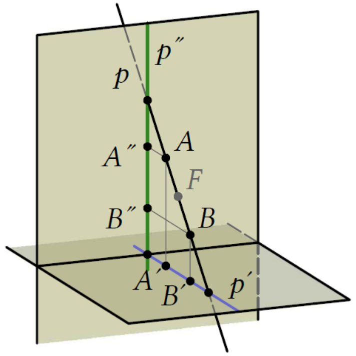
First/second projecting line – line v_1 / line v_2
(or horizontal/vertical projecting line)
Profile line – line p

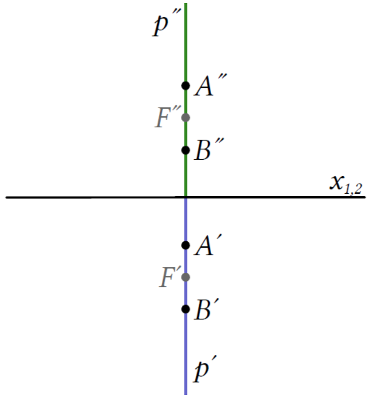












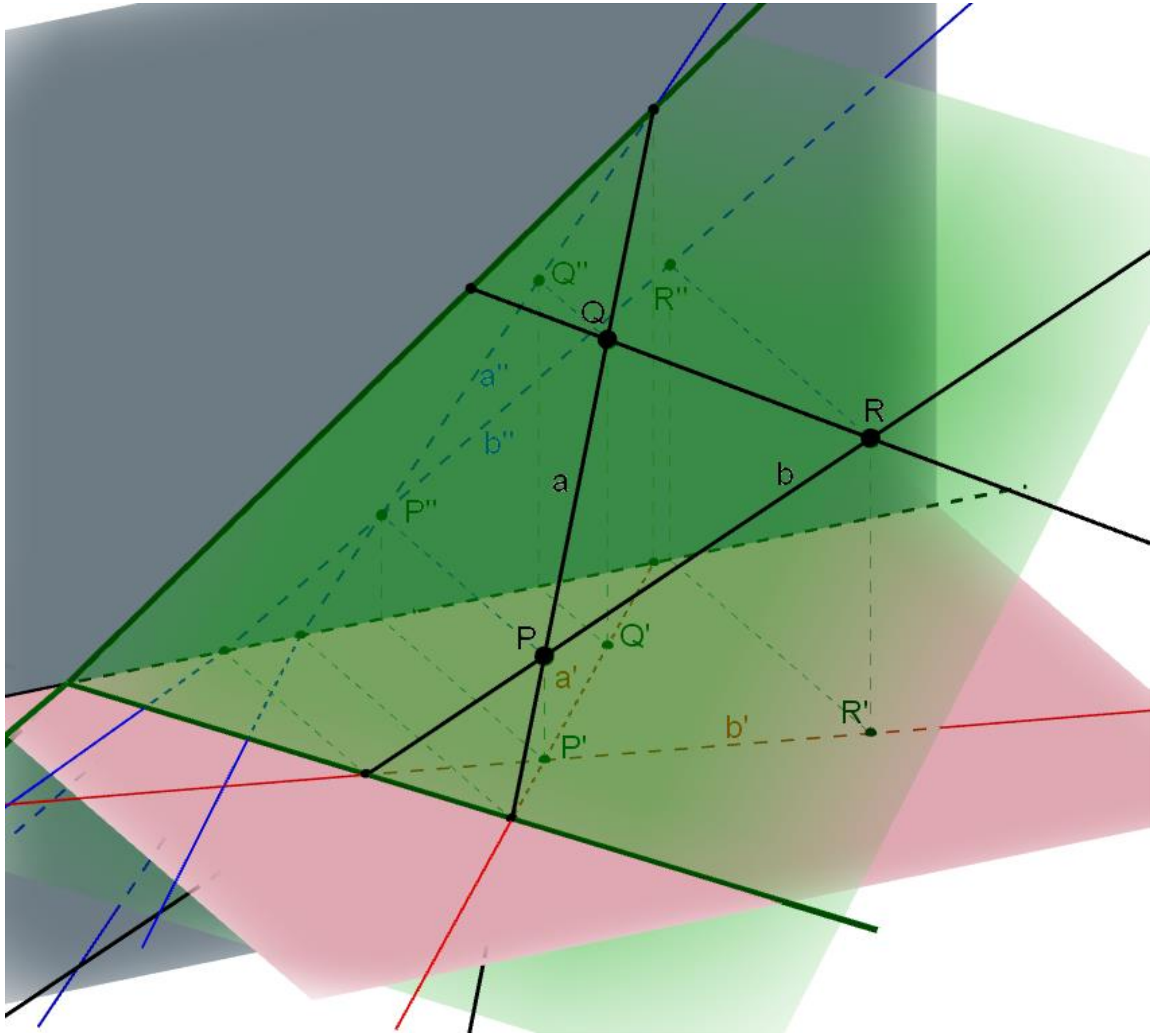
Source: Pék-Strommer – Ábrázoló geometria
(in Hungarian, for high school students)

Images of a plane – 1

A plane can be represented
by its three arbitrary points
(or its two arbitrary lines).

An arbitrary plane intersects the image planes.
The intersection lines of the given plane and
the image planes are called *the first and second tracing lines* (or horizontal and vertical trace), respectively.

(The tracing lines intersect each other on the axis.)



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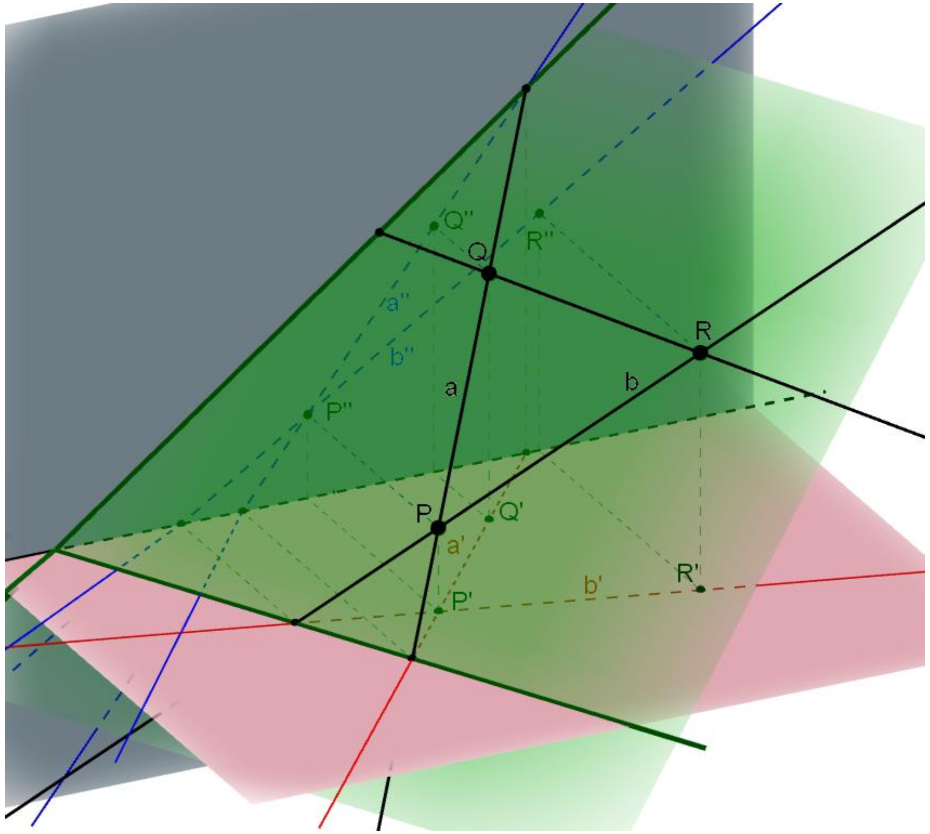
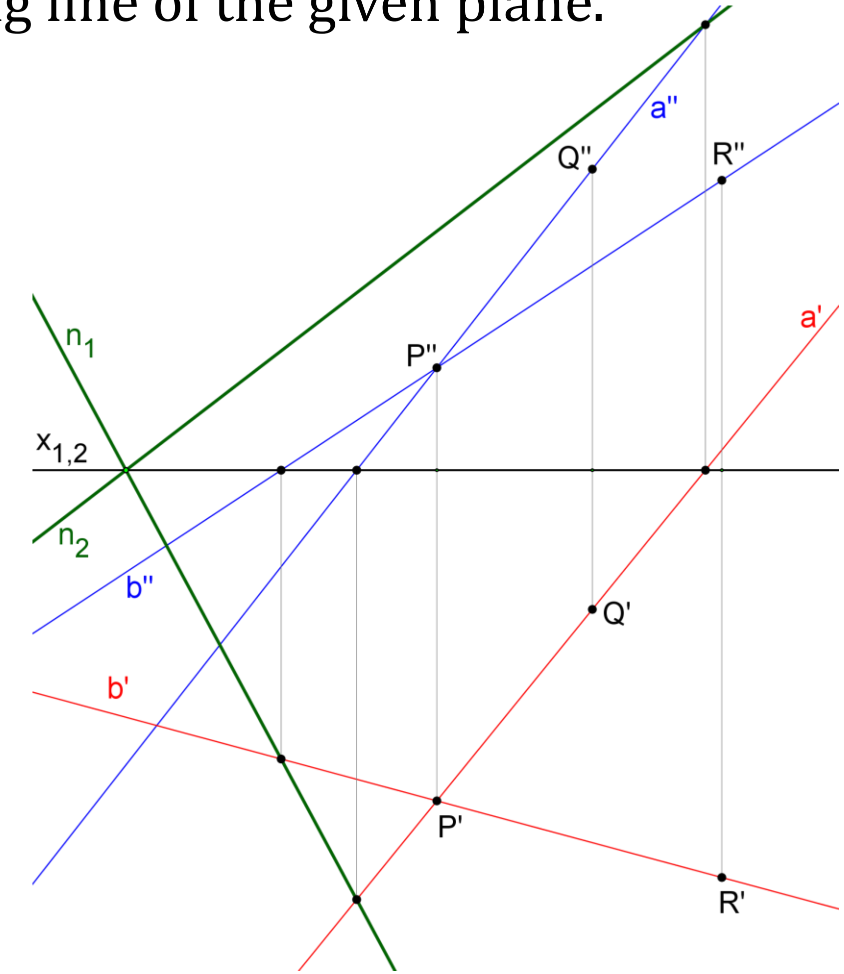
Images of a plane – 2

Remark: The tracing points of a line of the given plane lie on the tracing lines of the plane, respectively.
E.g. The first tracing point of line a lies on the first tracing line of the given plane.





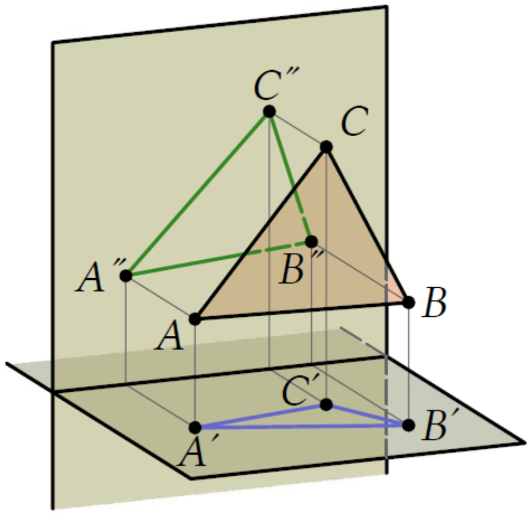
Slanted and spanned planes

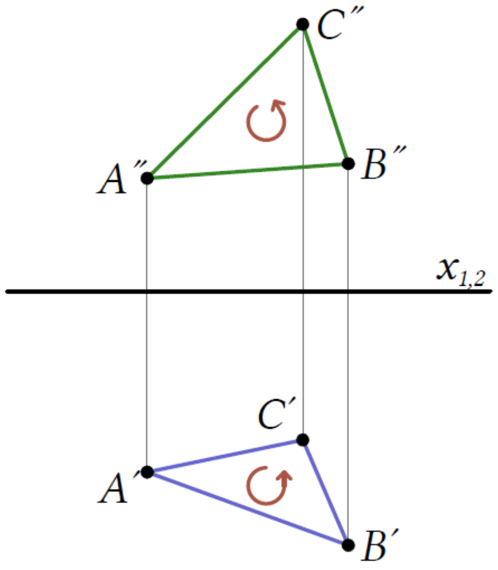
Slanted plane

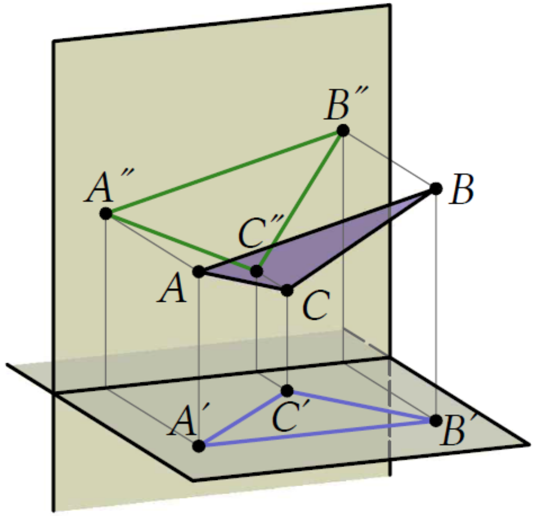
We can see the same side of the given plane.

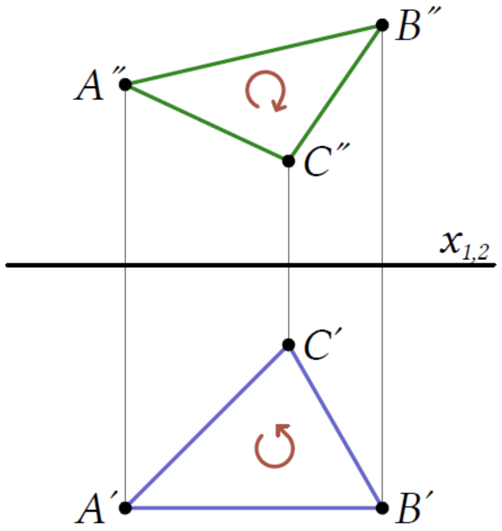
Spanned plane

We can see both sides of the given plane.









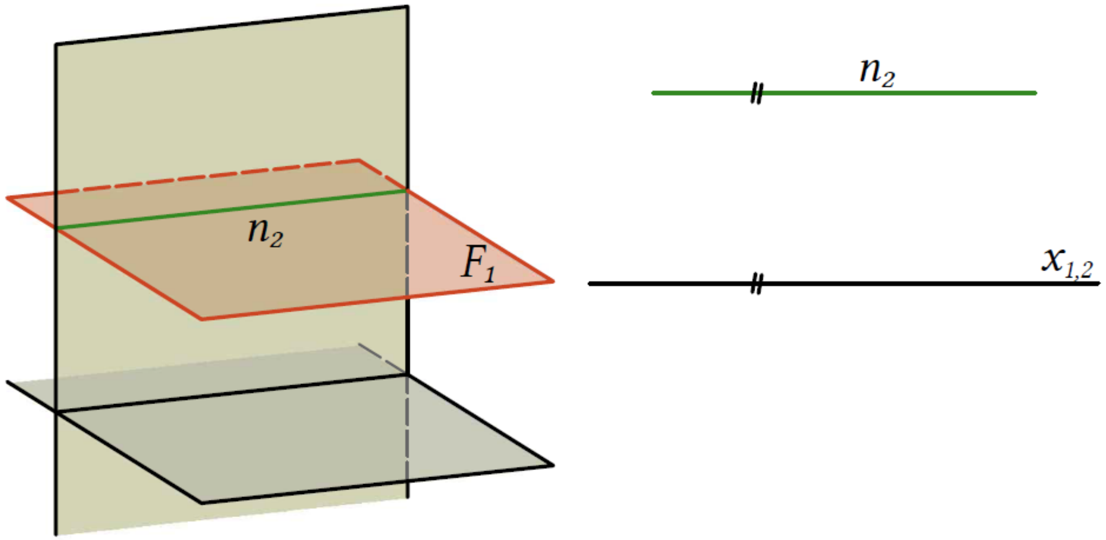
Remark: Pay attention to orientations of the images of the triangle.

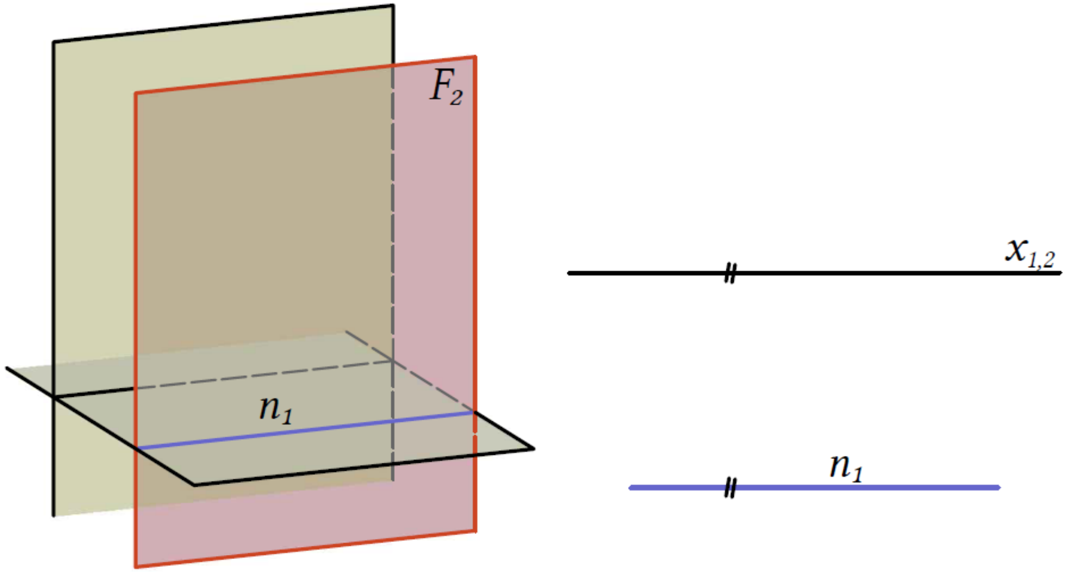
Source: Pék-Strommer – Ábrázoló geometria
(in Hungarian, for high school students)

Special planes – 1

Planes which are parallel to image planes

First/second principal plane – plane F_1 / plane F_2
(or horizontal/vertical plane or level/frontal plane)



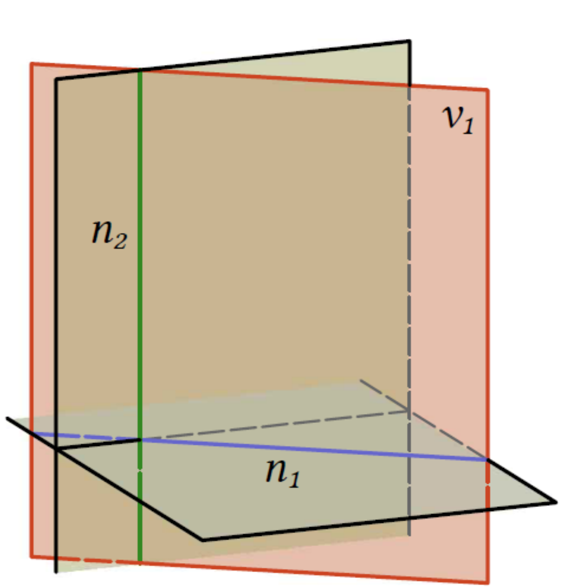


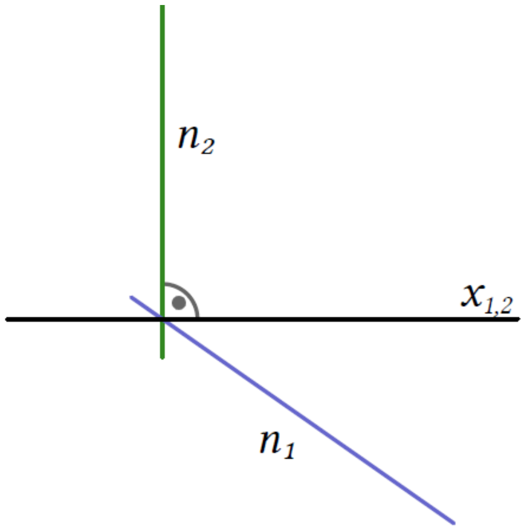
Source: Pék-Strommer – Ábrázoló geometria
(in Hungarian, for high school students)

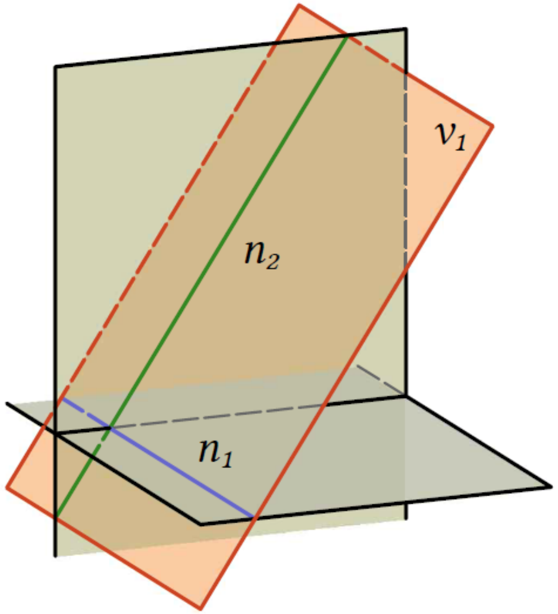
Special planes – 2

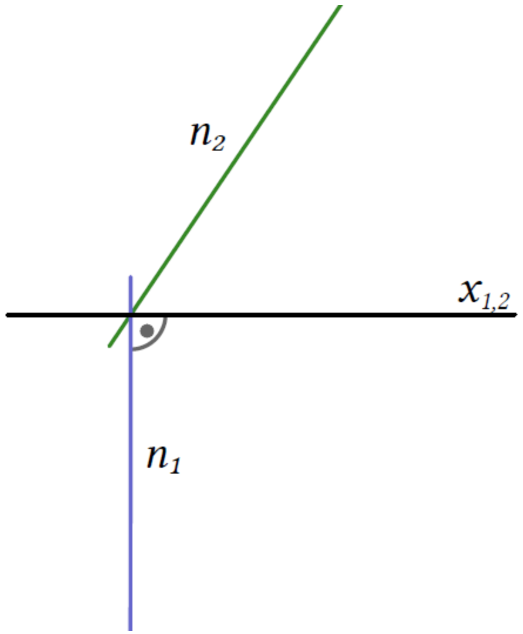
Planes which are perpendicular to image planes

First/second projecting plane – plane V_1 / plane V_2
(or horizontal/vertical projecting plane)







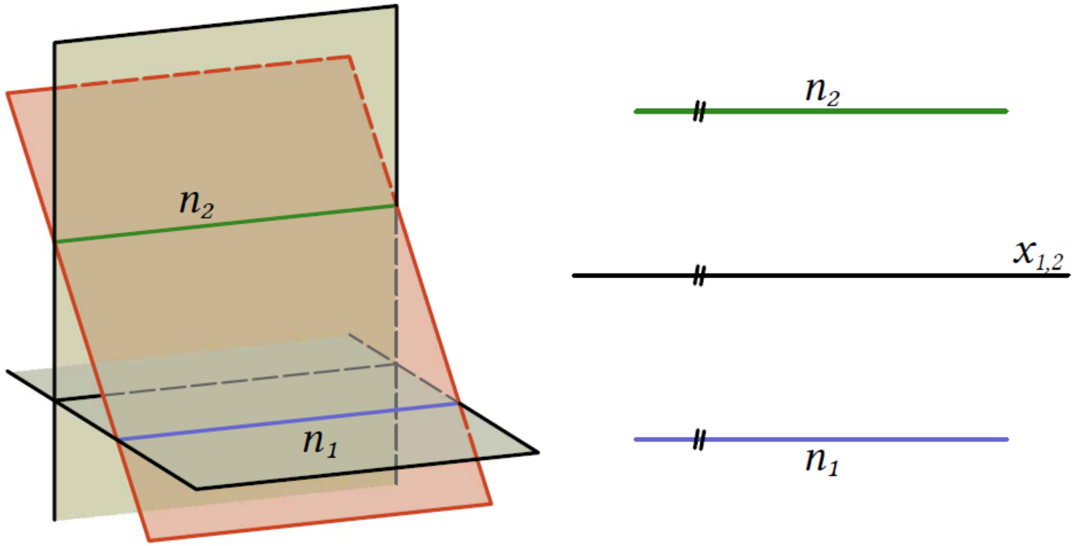


Source: Pék-Strommer – Ábrázoló geometria
(in Hungarian, for high school students)

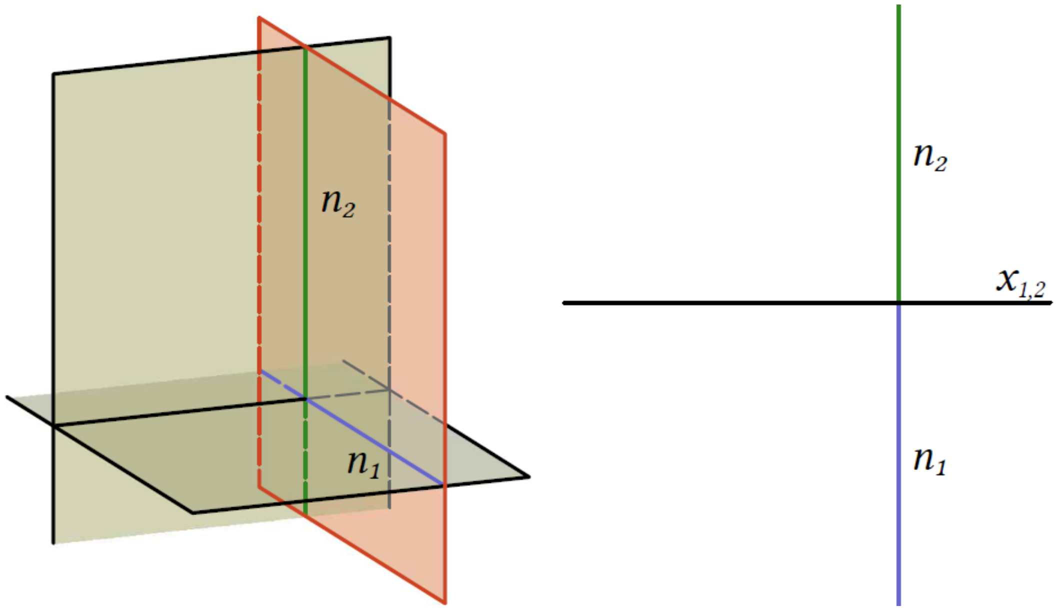
Special planes – 3

Planes which are parallel or perpendicular to the axis

Third projecting plane



Profile plane



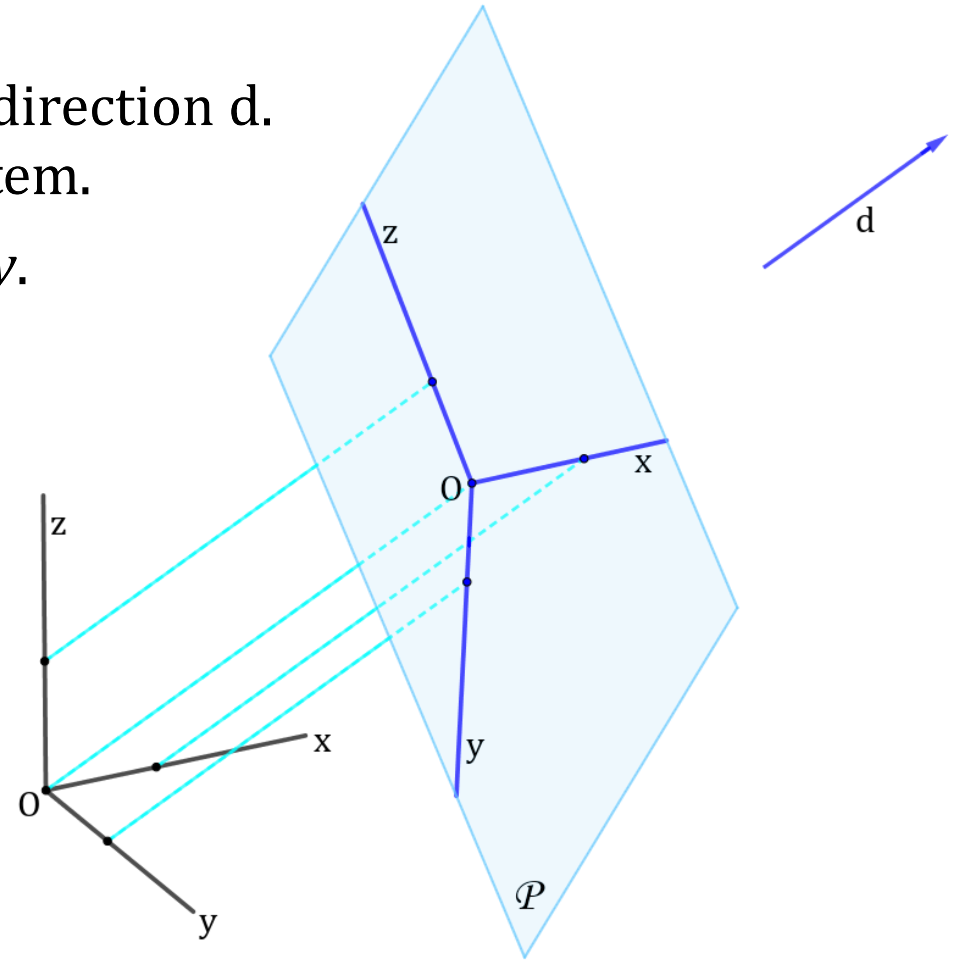
Source: Pék-Strommer – Ábrázoló geometria
(in Hungarian, for high school students)

AXONOMETRICAL REPRESENTATION

Consider the Cartesian coordinate system*, an arbitrary plane ($\mathcal{P}$) and a direction (d).

We can project the coordinate system onto plane $\mathcal{P}$ using direction d.
This image is *the axonometric image* of the coordinate system.

The method of this parallel projection is called *axonometry*.



$\mathcal{P}$ –(axonometric) image plane
d – direction of the projection

An axonometry is uniquely determined by the images of the origin and the unit points of the three coordinate axes.

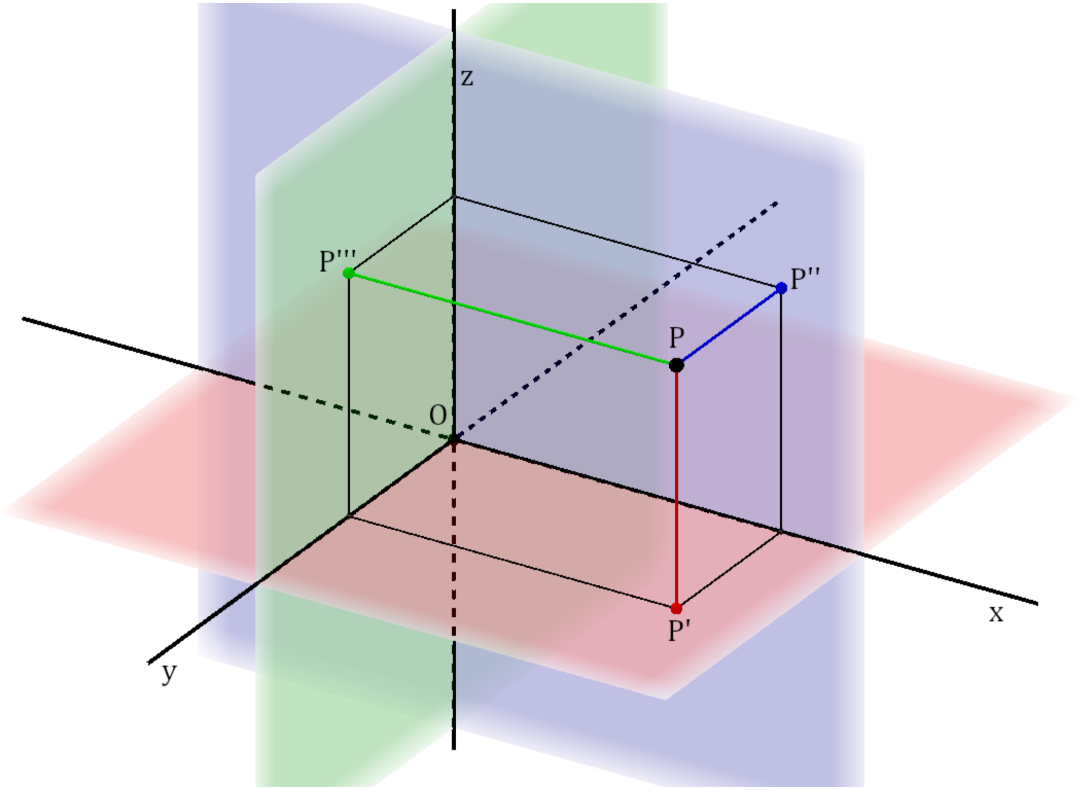
Every triplet of these four points must not lie on the same line.

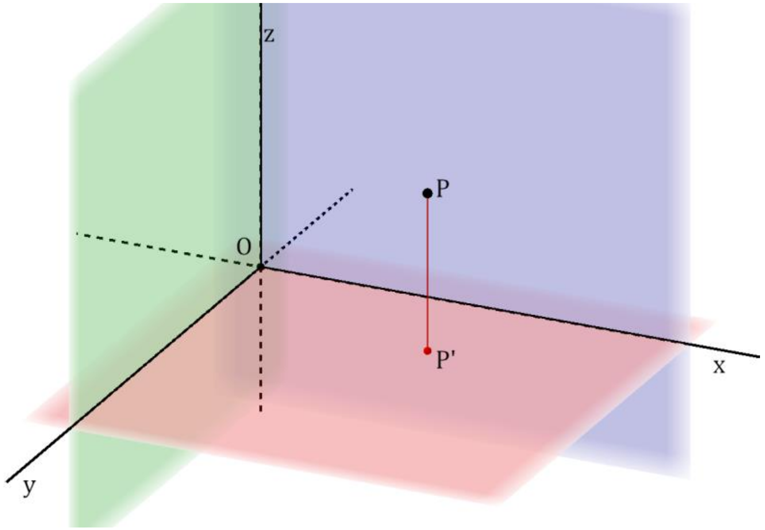
* left-handed coordinate system

Images of a point

Every object – e.g. a point – has four images:

P – *axonometric image*
P’ – *first image* (the axonometric image of the orthogonal projection of P on plane [x,y])
P’’ – *second image* (the axonometric image of the orthogonal projection of P on plane [x,z])
P’’’ – *third image* (the axonometric image of the orthogonal projection of P on plane [y,z])



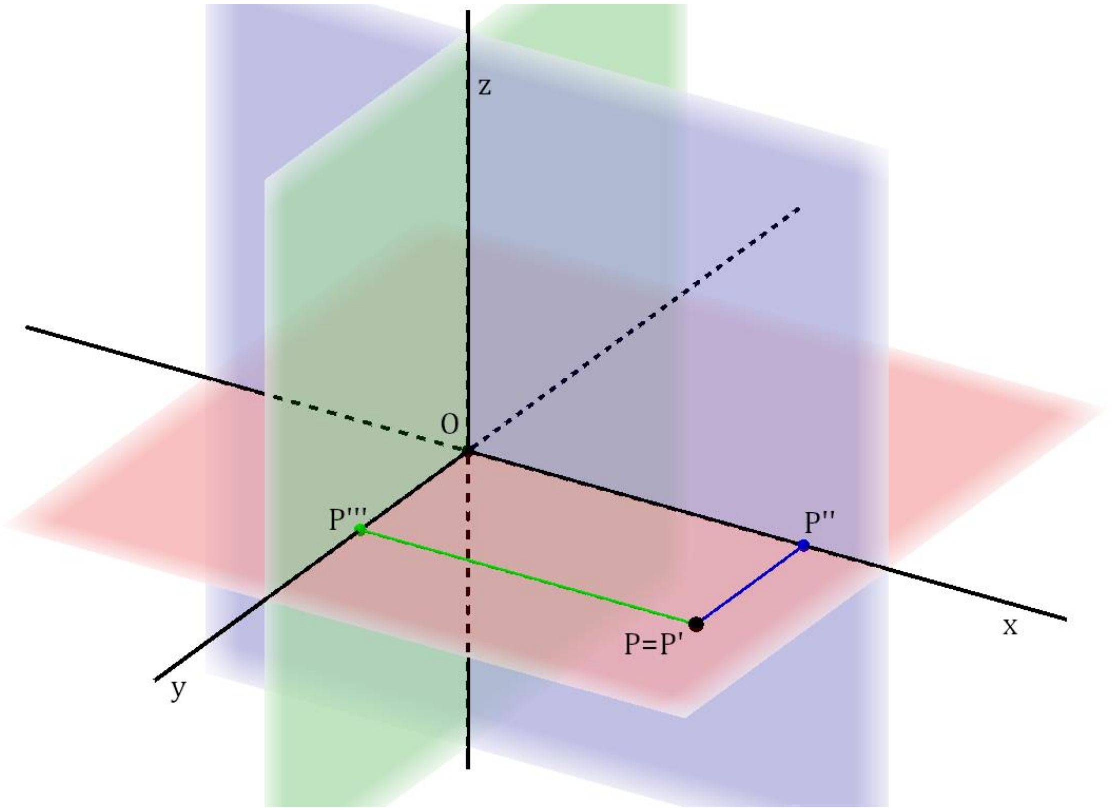


Every object – e.g. point P – is uniquely determined by two of its four images.

lecture / [Interactive version]

Special points

If a point lies on a coordinate plane then its axonometric image coincides with its first/second/third image.



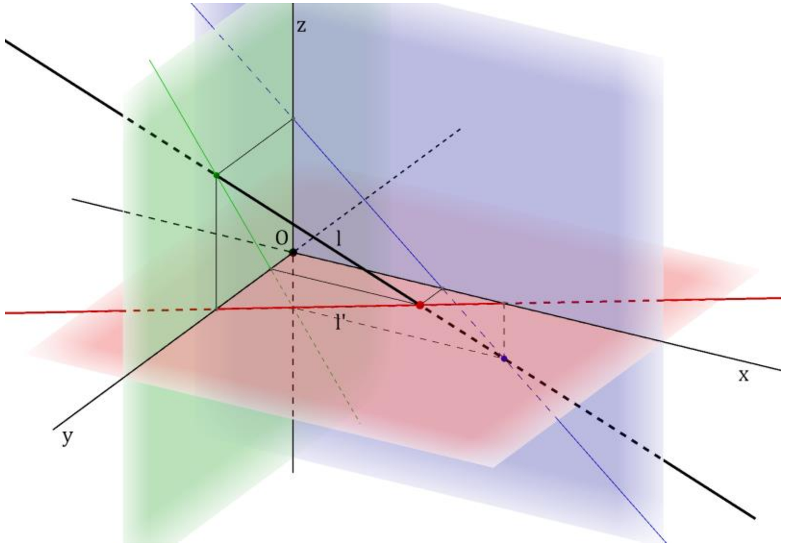
✍ lecture (P in [x,z] or [y,z])

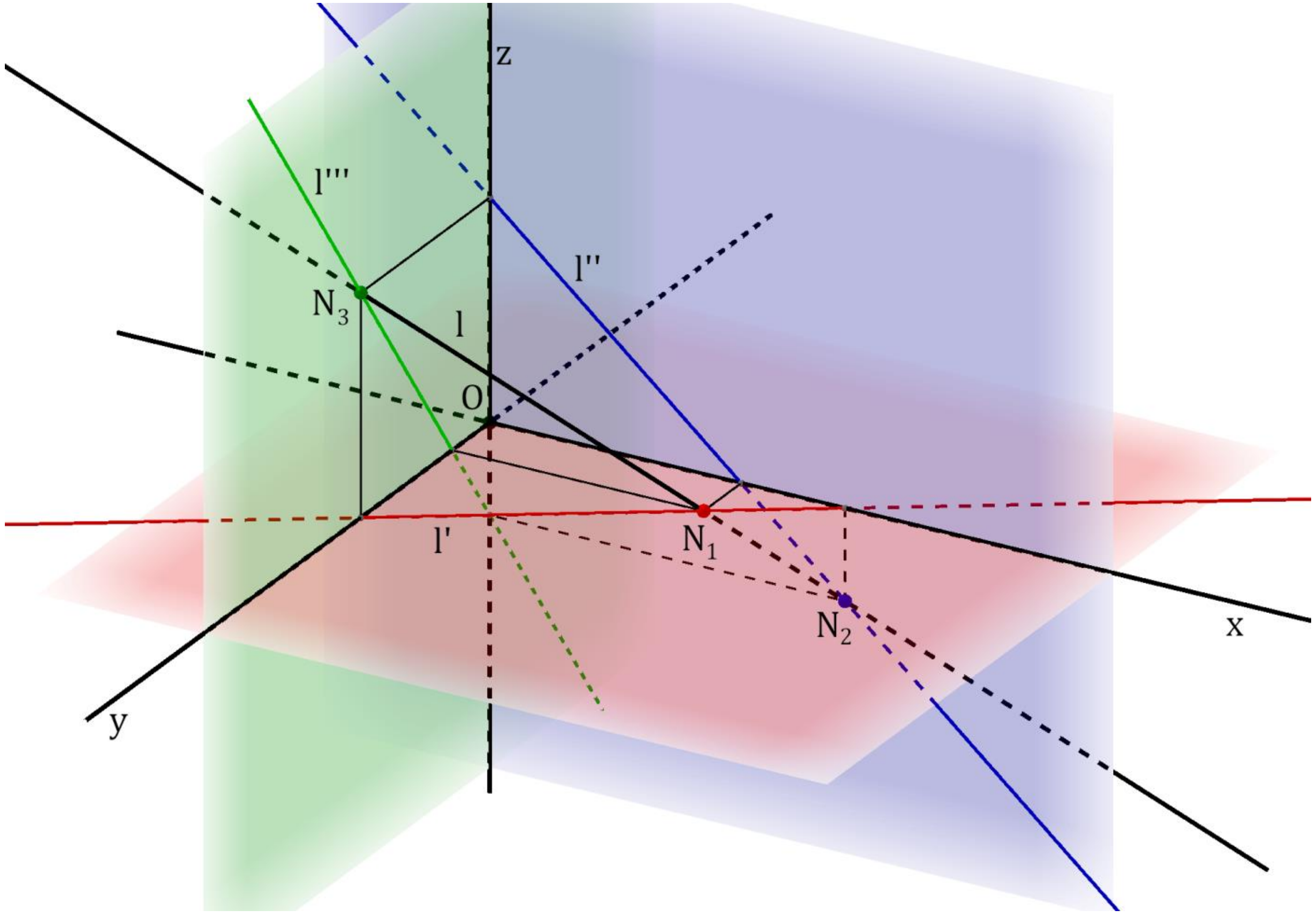
Images of a line

l – axonometric image
l', l'', l''' – first/second/third image

N_1, N_2, N_3 – first/second/third tracing point

Usually, a line is represented by its axonometric and first images.

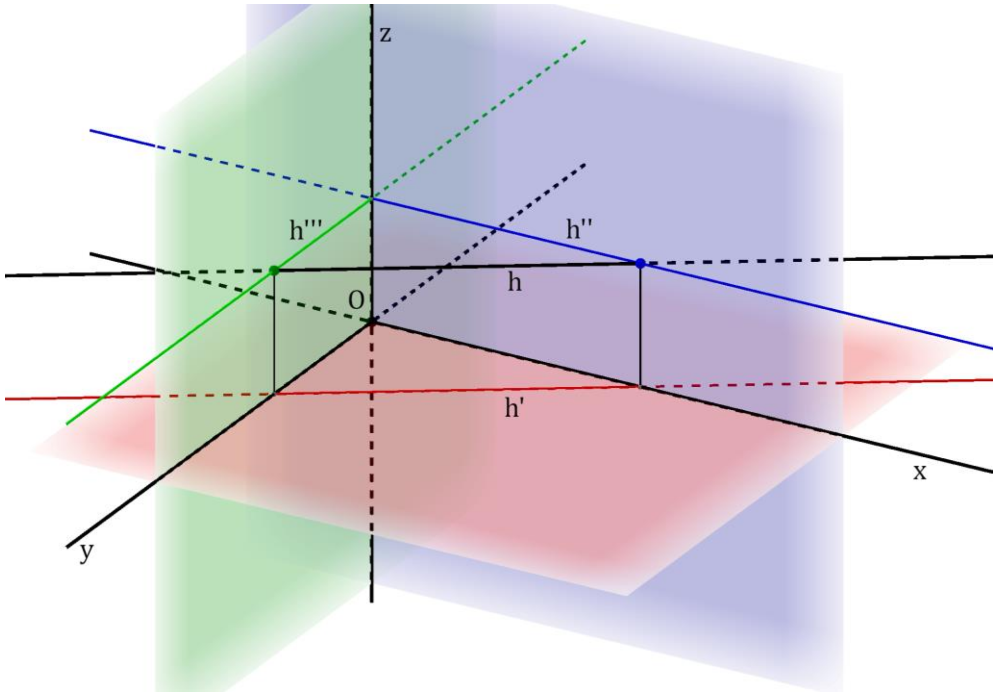




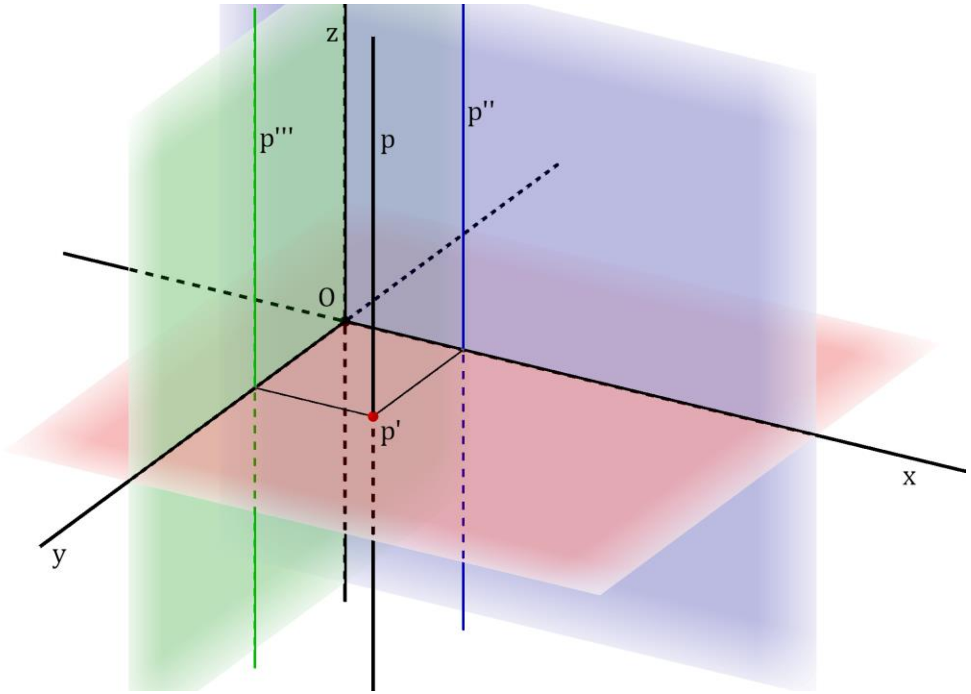
lecture (construction of the missing images) / [Interactive version]

Special lines – 1

A line which is parallel to a coordinate plane
→ *first principal line*



A line which is perpendicular to a coordinate plane → *first projecting line*



lecture (other examples)

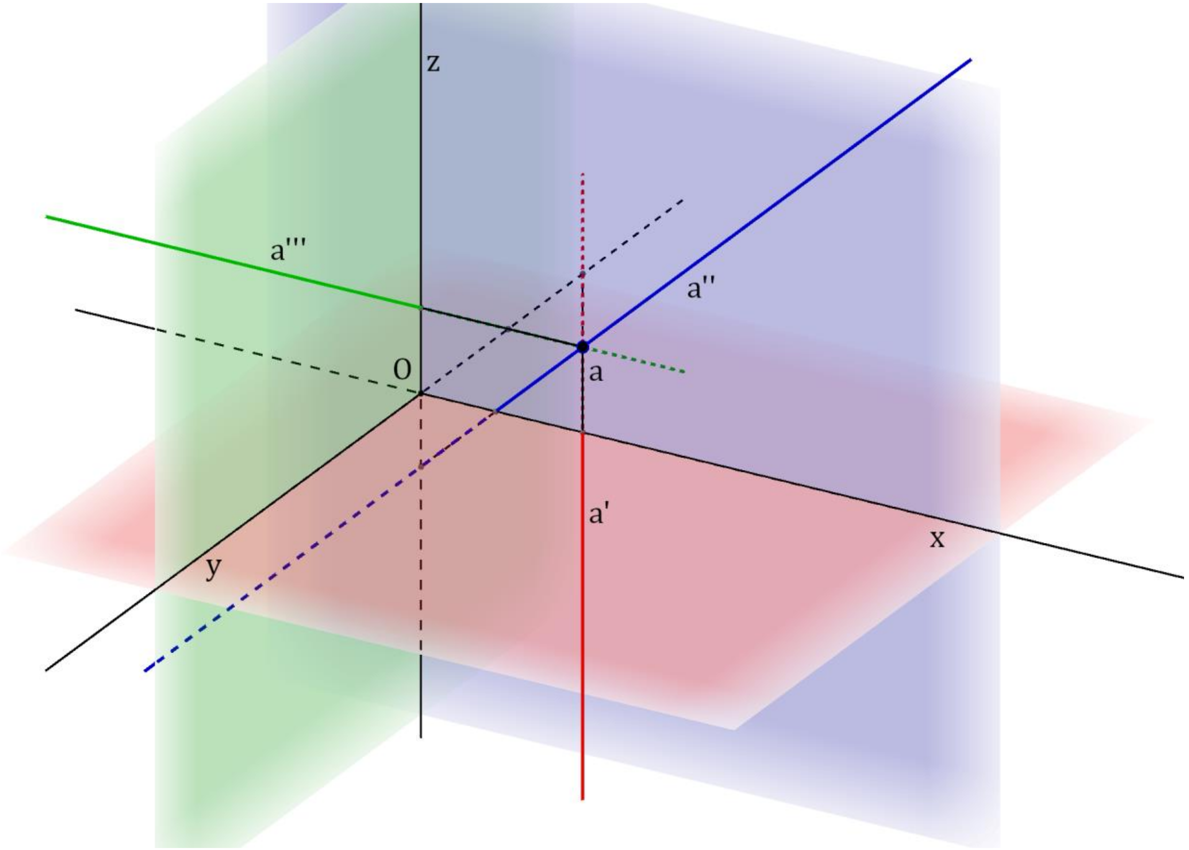
Special lines – 2

An interesting example: a line which is parallel to the direction of the projection.

It means that the axonometric image of the line is a single point.

After constructing the missing images, we obtain that its images seem to be parallel to the coordinate axes.
(In fact, they are not parallel to the axes.)

[Interactive version]



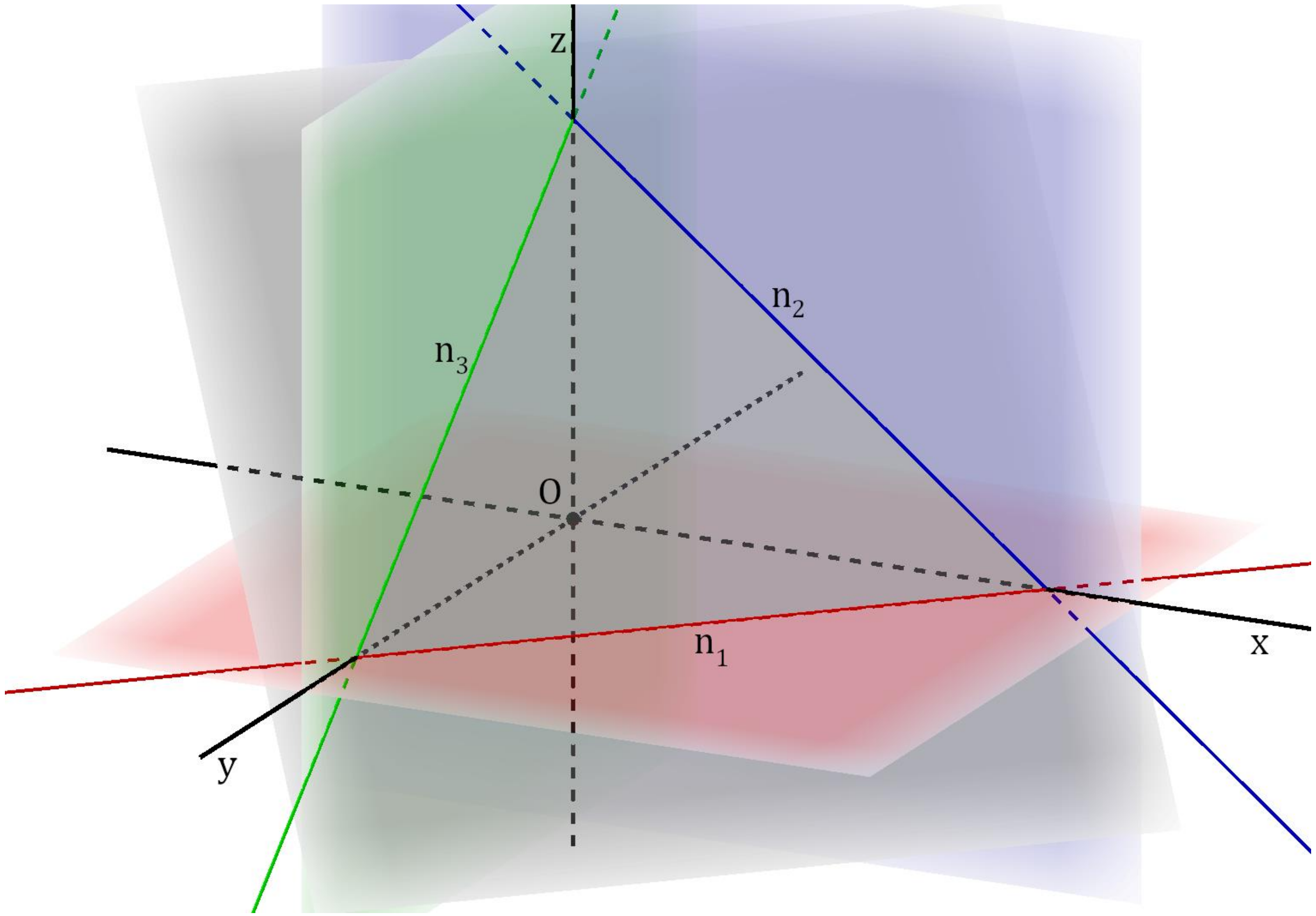
Images of a plane

A plane can be determined by the images of its three points or its two lines.

The most useful and easiest way to represent a plane is to give its intersection lines with the coordinate planes.

These lines are called *tracing lines*

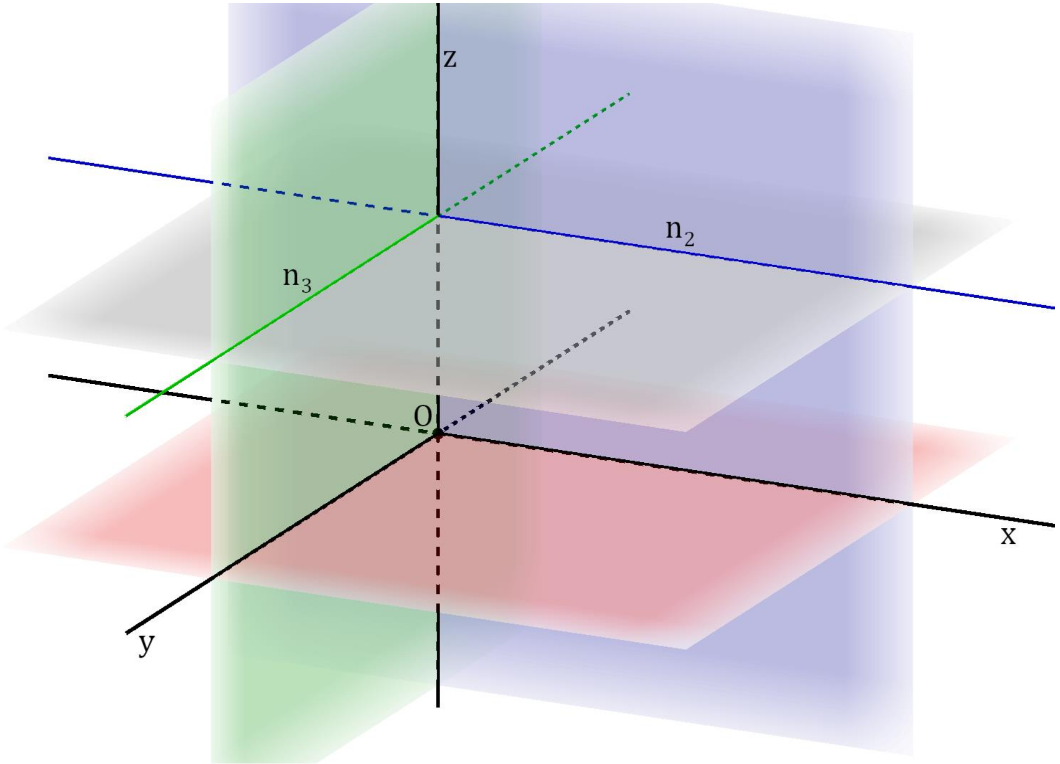
(denoted by n_1, n_2, n_3, respectively).



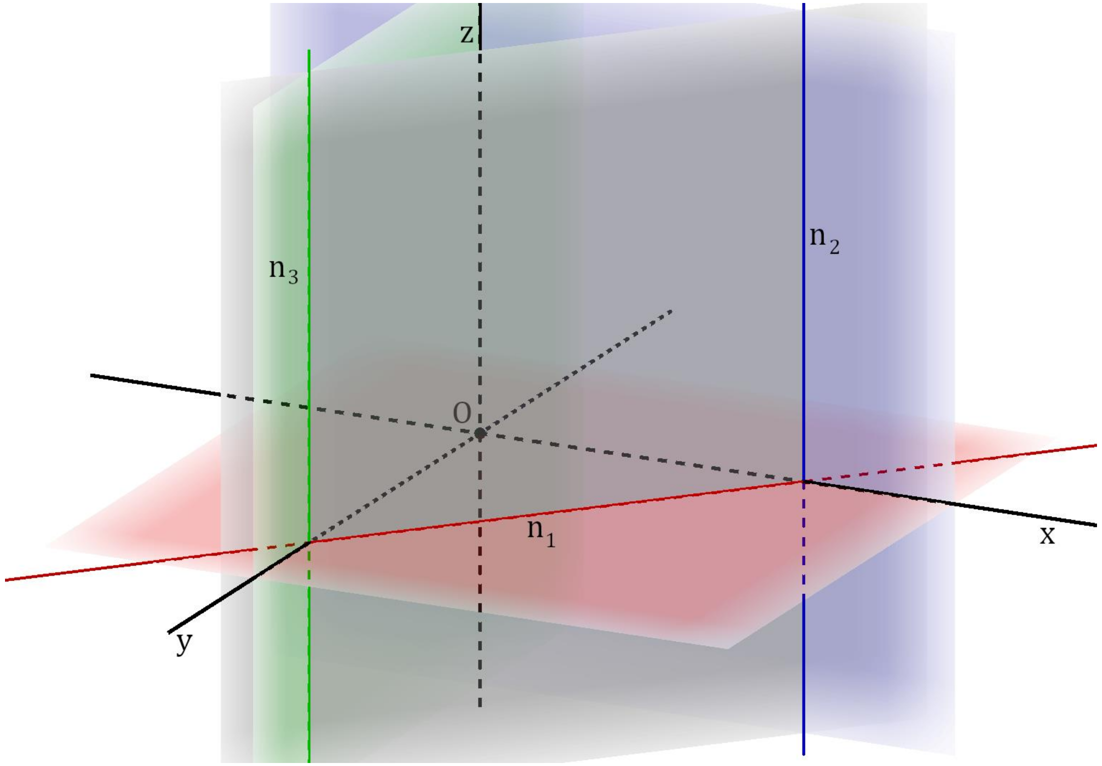
[Interactive version]

Special planes

A plane which is parallel to a coordinate plane (and perpendicular to an axis)
→ first principal plane



A plane which is perpendicular to a coordinate plane (and parallel to an axis)
→ first projecting plane



✍ lecture (other examples)