Automorphisms of line element D-manifolds

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Abstract. The main results of the paper generalize the following classical theorem to the setting of line element D-manifolds: the automorphisms of a covariant derivative on a manifold are exactly the affinities that leave its torsion invariant.

1 Introduction

In this paper, which is a continuation of our previous work [5], we study the automorphisms of so-called *line element D-manifolds*, i.e., structures consisting of a manifold M and a covariant derivative D on the pull-back of the tangent bundle $\tau\colon TM \to M$. The term was suggested by Serge LANG's terminology 'D-manifold' ([3], Ch. XIII). The covariant derivative we use was introduced by O. VARGA ([10]) and M. HASHIGUCHI ([2]), independently, in terms of classical tensor calculus. Line element D-manifolds provide a unified framework for a systematic study of covariant derivative operators appearing in Finsler geometry ([2], [7]).

The main results of the paper generalize the following well-known theorem: *the automorphisms of a covariant derivative on a manifold are exactly the affinities that leave its torsion invariant.*

Throughout the paper we use the coordinate-free calculus elaborated in [7] by J. SZILASI and apply the main results of our previous paper ([5]). These results are briefly summarized in section 3.

2 Preliminaries

As in [5], we follow the notation and conventions of [7] (see also [4] and [8]) as far as feasible. However, for the readers' convenience, in this section we fix some terminology and recall some basic facts.

'Manifold' will always mean a connected, second countable, Hausdorff, smooth manifold of dimension n, $n \geq 1$. If M is a manifold, $C^\infty(M)$ will denote the ring of smooth functions on M and $\mathrm{Diff}(M)$ the group of diffeomorphisms from M onto itself. $\tau\colon TM \to M$ (simply, τ or TM) is the tangent bundle of M. τ_{TM} denotes the canonical projection, the 'foot map', of TTM onto TM, as well as the tangent bundle of TM. If $\varphi\colon M \to N$ is a smooth map, then φ_* will denote the smooth map of TM into TN induced by φ, the tangent map or

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derivative of φ.

The vertical lift of a function $f \in C^\infty(M)$ is $f^{\mathsf{v}} := f \circ \tau$, the complete lift $f^{\mathsf{c}} \in C^\infty(TM)$ of f is defined by $f^{\mathsf{c}}(v) := v(f)$, $v \in TM$.
$\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of smooth vector fields on M. Any vector field X on M determines two vector fields on TM, the vertical lift X^{v} of X and the complete lift X^{c} of X, characterized by $X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}}$, $X^{\mathsf{v}} f^{\mathsf{v}} = 0$ and $X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}$, $X^{\mathsf{c}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}}$; $f \in C^\infty(M)$. It is easy to see that $[X^{\mathsf{v}}, Y^{\mathsf{v}}] = 0$ for all $X, Y \in \mathfrak{X}(M)$.

Throughout the paper, $I \subset \mathbb{R}$ will be an open interval. The velocity field of a smooth curve $\gamma \colon I \to M$ is $\dot\gamma := \gamma_* \circ \frac{d}{du} \colon I \to TM$, where $\frac{d}{du}$ is the canonical vector field on the real line. The acceleration field of γ is $\ddot\gamma = \left(\dot{\gamma}_* \circ \frac{d}{du}\right)_* \circ \frac{d}{du}$. If $\gamma \colon I \to M$ is a smooth curve and $\varphi \in \mathrm{Diff}(M)$, then we have $\dot{\overline{\varphi \circ \gamma}} = \varphi_* \circ \dot\gamma$, $\ddot{\overline{\varphi \circ \gamma}} = \varphi_{**} \circ \ddot\gamma$.

Let $\tau^*TM := TM \times_M TM := \{(u,v) \in TM \times TM \mid \tau(u) = \tau(v)\}$, and let $\tau^*\tau(u,v) := u$ for $(u,v) \in \tau^*TM$. Then $\tau^*\tau$ is a vector bundle with total space τ^*TM and base space TM, the *pull-back* of $\tau \colon TM \to M$ over τ. The $C^\infty(TM)$-module of sections of $\tau^*\tau$ will be denoted by $\mathrm{Sec}(\tau^*\tau)$. Any vector field X on M determines a section

$$\widehat{X} \colon v \in TM \longmapsto (v, X \circ \tau(v)) \in TM \times_M TM \ ,$$

called the *basic section* associated to X. $\mathrm{Sec}(\tau^*\tau)$ is generated by the basic sections. We have a *canonical section*

$$\delta \colon v \in TM \longmapsto (v, v) \in TM \times_M TM \ .$$

Generic sections in $\mathrm{Sec}(\tau^*\tau)$ will be denoted by $\widetilde{X}, \widetilde{Y}, \dots$.

Starting from the slit tangent bundle $\mathring{\tau} \colon \mathring{T}M \to M$, the pull-back bundle $\mathring{\tau}^*\tau \colon \mathring{T}M \times_M TM \to TM$ is constructed in the same way. Omitting the routine details, we remark that $\mathrm{Sec}(\tau^*\tau)$ may naturally be embedded into the $C^\infty(\mathring{T}M)$-module $\mathrm{Sec}(\mathring{\tau}^*\tau)$.
There exists a canonical injective bundle map $\mathbf{i} \colon TM \times_M TM \to TTM$ given by

$$\mathbf{i}(u,v) := \dot c(0) \ , \qquad \text{if} \quad c(t) := u + tv \quad (t \in \mathbb{R}) \ ,$$

and a canonical surjective bundle map

$$\mathbf{j} \colon TTM \to TM \times_M TM \ ,$$
$$w \in T_v TM \longmapsto \mathbf{j}(w) := (v, \tau_*(w)) \in \{v\} \times T_{\tau(v)} M \ .$$

Then $\mathbf{j} \circ \mathbf{i} = 0$, while $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is a further important canonical object, the *vertical endomorphism* of TTM. $\mathbf{i}$ and $\mathbf{j}$ induce the tensorial maps

$$\widetilde{X} \in \mathrm{Sec}(\tau^*\tau) \longmapsto \mathbf{i}\widetilde{X} := \mathbf{i} \circ \widetilde{X} \in \mathfrak{X}(TM) \qquad \text{and}$$
$$\xi \in \mathfrak{X}(TM) \longmapsto \mathbf{j}\xi := \mathbf{j} \circ \xi \in \mathrm{Sec}(\tau^*\tau) \ ,$$

so $\mathbf{J}$ may also be interpreted as a $C^\infty(TM)$-linear endomorphism of $\mathfrak{X}(TM)$. $\mathfrak{X}^{\mathsf{v}}(TM) := \mathbf{i}\mathrm{Sec}(\tau^*\tau)$ is the module of *vertical vector fields* on TM. The vertical vector fields form a subalgebra of the Lie algebra $\mathfrak{X}(TM)$ at the same time. For any vector field X on M we have $\mathbf{i}\widehat{X} = X^{\mathsf{v}}$ and $\mathbf{j}X^{\mathsf{c}} = \widehat{X}$. $C := \mathbf{i}\delta$ is a canonical vertical vector field, called the *Liouville vector field* on TM. If $\gamma\colon I \to M$ is a smooth curve, then

$$\mathbf{j} \circ \ddot{\gamma} = \delta \circ \dot{\gamma}. \tag{1}$$

Recall that the push-forward of a vector field $X \in \mathfrak{X}(M)$ or a vector field $\xi \in \mathfrak{X}(TM)$ or a section $\widetilde{X} \in \mathrm{Sec}(\tau^*\tau)$ by a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ is the vector field (or the section)

$$\varphi_\# X := \varphi_* \circ X \circ \varphi^{-1}\ ; \qquad (\varphi_*)_\# \xi := \varphi_{**} \circ \xi \circ (\varphi_*)^{-1}\ ;$$
$$\varphi_\# \widetilde{X} := (\varphi_* \times \varphi_*) \circ \widetilde{X} \circ \varphi_*^{-1}\ ,$$

where $\varphi_* \times \varphi_*\colon\ (u,v) \in TM \times_M TM \longmapsto (\varphi_*(u), \varphi_*(v)) \in TM \times_M TM$.
It follows at once that

$$\varphi_\# \delta = \delta\ , \qquad \varphi_\# \widehat{X} = \widehat{\varphi_\# X}\ , \quad (X \in \mathfrak{X}(M)). \tag{2}$$

We also have

$$(\varphi_*)_\# \circ \mathbf{i} = \mathbf{i} \circ \varphi_\#\ , \quad \varphi_\# \circ \mathbf{j} = \mathbf{j} \circ (\varphi_*)_\#\ , \quad (\varphi_*)_\# \circ \mathbf{J} = \mathbf{J} \circ (\varphi_*)_\#\ ;$$

and for any vector field X on M,

$$(\varphi_*)_\# X^{\mathsf{c}} = (\varphi_\# X)^{\mathsf{c}}\ , \quad (\varphi_*)_\# X^{\mathsf{v}} = (\varphi_\# X)^{\mathsf{v}}\ .$$

3 Semisprays and Ehresmann connections

A map $S\colon TM \to TTM$, smooth on $\mathring{T}M$, is said to be a *semispray*, if $\tau_{TM} \circ S = 1_{TM}$, it sends zeros to zeros, and satisfies the condition $\mathbf{j}S = \delta$ (or, equivalently, $\mathbf{J}S = C$). By a *spray* we mean a semispray of class C^1, which is positive-homogeneous of degree two in the sense that $[C, S] = S$.
A regular curve $\gamma\colon I \to M$ is a *geodesic* of a semispray S if its velocity field is an integral curve of S, i.e., $S \circ \dot{\gamma} = \ddot{\gamma}$. A diffeomorphism $\varphi\colon M \to M$ is an *affinity* (or *totally geodesic transformation*) of S if it preserves the geodesics considered as parametrized curves, i.e., if

$$\ddot{\overline{\varphi \circ \gamma}} = S \circ \dot{\overline{\varphi \circ \gamma}}, \qquad \text{for all geodesics } \gamma\colon I \to M\ .$$

The affinities of a semispray S form a Lie group, denoted by $\mathrm{Aff}(S)$.
If S is a semispray and $\varphi \in \mathrm{Diff}(M)$, then $(\varphi_*)_\# S$ is also a semispray, which remains a spray, if S is a spray. φ is called an *automorphism of* S, if $(\varphi_*)_\# S = S$, i.e., $\varphi_{**} \circ S = S \circ \varphi_*$. $\mathrm{Aut}(S)$ denotes the group of automorphisms of S.

Lemma 3.1 ([5] Lemma 5.1) *The automorphism group of a semispray coincides with the group of affinities of the semispray.*

Roughly speaking, an *Ehresmann connection* $\mathcal{H}$ over a manifold M is a right splitting of the canonical exact sequence

$$0 \longrightarrow TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \longrightarrow 0 \ ,$$

smooth only on $\overset{\circ}{T}M \times_M TM$, and given on $o(M) \times_M TM$ by $\mathcal{H}(o(p), v) := (o_*)_p(v)$; $p \in M$, $v \in T_pM$, where $o \in \mathfrak{X}(M)$ is the zero vector field. We associate to any Ehresmann connection $\mathcal{H}$ the *horizontal projector* $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$, the *vertical projector* $\mathbf{v} = 1_{TTM} - \mathbf{h}$, the *vertical map* $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}$ and the *semispray* $S_{\mathcal{H}} := \mathcal{H} \circ \delta$. The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ with respect to $\mathcal{H}$ is $X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \mathbf{h}X^{\mathsf{c}} \in \mathfrak{X}(\overset{\circ}{T}M)$.
A regular smooth curve $\gamma \colon I \to M$ is a *geodesic* of an Ehresmann connection $\mathcal{H}$ if $\mathcal{V} \circ \ddot{\gamma} = 0$ or, equivalently, if $\ddot{\gamma}(t) \in \operatorname{Im}(\mathcal{H})$ $(t \in I)$, i.e., if the acceleration vector field of γ is horizontal with respect to $\mathcal{H}$.
If M is a manifold with an Ehresmann connection $\mathcal{H}$, then a diffeomorphism of M is said to be an *affinity* (*affine collineation*, or, by J. Vilms's terminology [11], a *totally geodesic map*) if it preserves the geodesics considered as parametrized curves. We denote by $\operatorname{Aff}(\mathcal{H})$ the group of these transformations.

Lemma 3.2 ([5] Lemma 6.1) *If M is a manifold with an Ehresmann connection $\mathcal{H}$, then* $\operatorname{Aff}(\mathcal{H}) = \operatorname{Aff}(S_{\mathcal{H}})$.

An Ehresmann connection $\mathcal{H}$ determines a covariant derivative operator ∇ in the pull-back bundle $\tau^*\tau$ by the rule

$$\nabla_\xi \widetilde{Y} := \mathbf{j}\left[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}\right] + \mathcal{V}\left[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}\right] \ ; \qquad \xi \in \mathfrak{X}(TM), \widetilde{Y} \in \operatorname{Sec}(\tau^*\tau) \ .$$

∇ is said to be the *Berwald derivative* induced by $\mathcal{H}$. For any vector fields $\widetilde{X}, \widetilde{Y} \in \operatorname{Sec}(\tau^*\tau)$, the *v-part* ∇^{v} and the *h-part* ∇^{h} of the Berwald derivative are defined by

$$\nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y} := \nabla_{\mathbf{i}\widetilde{X}} \widetilde{Y} = \mathbf{j}\left[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}\right] \quad \text{and} \quad \nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} := \nabla_{\mathcal{H}\widetilde{X}} \widetilde{Y} = \mathcal{V}\left[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}\right] \ .$$

$\mathbf{t} := \nabla^{\mathsf{h}}\delta$, $\mathbf{T}(\widetilde{X}, \widetilde{Y}) = \nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} - \nabla^{\mathsf{h}}_{\widetilde{Y}} \widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]$ $(\widetilde{X}, \widetilde{Y} \in \operatorname{Sec}(\overset{\circ}{\tau}{}^*\tau))$ and $\mathbf{T}^{\mathsf{s}} := \mathbf{t} + i_\delta \mathbf{T}$ are the *tension*, the *torsion*, and the *strong torsion* of $\mathcal{H}$, respectively. $\mathcal{H}$ is called *homogeneous* if its tension vanishes. In the homogeneous case the associated semispray $S_{\mathcal{H}}$ is a spray.

$\varphi^{\#}\mathcal{H} := \varphi_{**}^{-1} \circ \mathcal{H} \circ (\varphi_* \times \varphi_*)$ is said to be the *pull-back of $\mathcal{H}$ by φ*. If $\varphi^{\#}\mathcal{H} = \mathcal{H}$, i.e., $\varphi_{**} \circ \mathcal{H} = \mathcal{H} \circ (\varphi_* \times \varphi_*)$, then φ is called an *automorphism* of $\mathcal{H}$.

Theorem 3.3 ([5] Theorem 7.5) *A diffeomorphism φ of M is an automorphism of an Ehresmann connection $\mathcal{H}$ over M if and only if it is an automorphism of the associated semispray $S_{\mathcal{H}}$, and $\varphi_{\#} \circ \mathbf{T}^{\mathsf{s}} = \mathbf{T}^{\mathsf{s}} \circ \varphi_{\#}$.*

Corollary 3.4 ([5] Cor. 7.6) *If M is a manifold with an Ehresmann connection $\mathcal{H}$, then a diffeomorphism φ of M is an automorphism of $\mathcal{H}$ if and only if φ is an affinity, and $\varphi_{\#}$ commutes with the strong torsion of $\mathcal{H}$.*

4 Line element D-manifolds

By a *line element D-manifold* we mean a pair (M, D) consisting of a manifold M and a covariant derivative D in pull-back bundle $\tau^*\tau$. The *v-covariant derivative* D^{v} belonging to D is given by $D^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} = D_{\mathbf{i}\widetilde{X}}\widetilde{Y}$ $(\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau))$. The *torsion* $T(D)$ of D is defined by

$$T(D)(\xi, \eta) := D_\xi \mathbf{j}\eta - D_\eta \mathbf{j}\xi - \mathbf{j}[\xi, \eta]; \qquad \xi, \eta \in \mathfrak{X}(TM).$$

The *vertical difference tensor* $\mathcal{S}$ of D is given by

$$\mathcal{S}(\widetilde{X}, \widetilde{Y}) := \nabla^{\mathsf{v}}_{\widetilde{Y}}\widetilde{X} - D^{\mathsf{v}}_{\widetilde{Y}}\widetilde{X} = \mathbf{j}[\mathbf{i}\widetilde{Y}, \eta] - D_{\mathbf{i}\widetilde{Y}}\widetilde{X} \ ; \qquad (\mathbf{j}\eta = \widetilde{X})$$

$(\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau),\ \eta \in \mathfrak{X}(TM))$, it is also mentioned (see [1]) as the *Finsler torsion* of D.

Let D be a covariant derivative in $\tau^*\tau$. The $\binom{1}{1}$ tensors $\mu := D\delta$ and $\mu^{\mathsf{v}} := \mu \circ \mathbf{i}$ are said to be the *deflection* and the *v-deflection* of D, respectively. We say that D is *regular*, if μ^{v} is fibrewise injective; *strongly regular*, if $\mu^{\mathsf{v}} = 1_{\mathrm{Sec}(\tau^*\tau)}$. If an Ehresmann connection $\mathcal{H}$ over M is also given, then we define the *h-covariant derivative* D^{h} by $D^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} := D_{\mathcal{H}\widetilde{X}}\widetilde{Y}$ for all $\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$. Then the $\binom{1}{1}$ tensor $\mu^{\mathcal{H}} := D^{\mathsf{h}}\delta = \mu \circ \mathcal{H}$ is said to be the *$\mathcal{H}$-deflection of D*.

Theorem and Definition 4.1 ([4] Prop. 3) *If D is a* regular *covariant derivative in $\tau^*\tau$, then there is a unique Ehresmann connection $\mathcal{H}_D$ over M such that the $\mathcal{H}_D$-deflection of D vanishes, and hence* $\mathrm{Ker}(\mu) = \mathrm{Im}(\mathcal{H}_D)$. *On basic vector fields $\mathcal{H}_D$ acts by*

$$\mathcal{H}_D(\widehat{X}) = X^{\mathsf{c}} - \mathbf{i}(\mu^{\mathsf{v}})^{-1} D_{X^{\mathsf{c}}}\delta \ , \qquad X \in \mathfrak{X}(M) \ .$$

If $\mathcal{H}_D$ is the Ehresmann connection induced by D, then we can define the *horizontal torsion* $\mathcal{T}$, the *vertical torsion* $T^{\mathsf{v}}(D)$, and the *horizontal difference tensor* $\mathcal{P}$ as follows: for each $\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$; $\xi, \eta \in \mathfrak{X}(TM)$,

(3) $$\mathcal{T}(\widetilde{X}, \widetilde{Y}) := D_{\mathcal{H}_D\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}_D\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}_D\widetilde{X}, \mathcal{H}_D\widetilde{Y}],$$

(4) $$T^{\mathsf{v}}(D)(\xi, \eta) := D_\xi \mathcal{V}_D\eta - D_\eta \mathcal{V}_D\xi - \mathcal{V}_D[\xi, \eta],$$

(5) $$\mathcal{P}(\widetilde{X}, \widetilde{Y}) := D_{\mathcal{H}_D\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}_D\widetilde{X}}\widetilde{Y}.$$

Then we have

(6) $$\mathcal{S}(\widetilde{X}, \widetilde{Y}) = T(D)(\mathcal{H}_D\widetilde{X}, \mathbf{i}\widetilde{Y}), \quad \widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau) \ ,$$

(7) $$\mathbf{T}^{\mathbf{s}} = i_\delta\mathcal{T} - i_\delta\mathcal{P} \ ,$$

(8) $$\mathcal{P}(\widetilde{X}, \widetilde{Y}) = T^{\mathsf{v}}(D)(\mathcal{H}_D\widetilde{X}, \mathbf{i}\widetilde{Y}) \ ,$$

where $\mathbf{T}^{\mathbf{s}}$ is the strong torsion of $\mathcal{H}_D$.

5 Affinities and automorphisms of line element D-manifolds

A regular smooth curve $\gamma\colon I \to M$ is said to be a *geodesic* of a line element D-manifold (M, D) if $D_{\dot\gamma}(\delta \circ \dot\gamma) = 0$, or equivalently (see (1))

$$D_{\dot\gamma}(\mathbf{j} \circ \ddot\gamma) = 0. \tag{9}$$

A diffeomorphism φ is an *affinity* (or a *totally geodesic transformation*) of (M, D) if for any geodesic $\gamma\colon I \longrightarrow M$, $\varphi \circ \gamma$ is also a geodesic. The group of affinities of D is denoted by $\mathrm{Aff}(D)$.

Lemma 5.1 *Let (M, D) be a regular line element D-manifold and $\mathcal{H}_D$ be the Ehresmann connection induced by D. Then the geodesics of D coincide with geodesics of $\mathcal{H}_D$, therefore we have*

$$\mathrm{Aff}(D) = \mathrm{Aff}(\mathcal{H}_D). \tag{10}$$

Proof. Let $\gamma\colon I \to M$ be a geodesic of D. Then we have for all $t \in I$

$$D_{\ddot\gamma}(\delta \circ \dot\gamma)(t) = 0 \overset{\text{def}}{\iff} D_{\ddot\gamma(t)}\delta = 0 \iff D\delta(\ddot\gamma(t)) = 0 \iff$$
$$\iff \ddot\gamma(t) \in \mathrm{Ker}(D\delta) \overset{4.1}{\iff} \ddot\gamma(t) \in \mathrm{Im}(\mathcal{H}_D)\ ,$$

so D and $\mathcal{H}_D$ have the same geodesics. □

If $\varphi \in \mathrm{Diff}(M)$ then $(\varphi_* \times \varphi_*, \varphi_*)$ is an automorphism of the pull-back bundle $\tau^*\tau$. So we may consider the pull-back of covariant derivative D of a line element D-manifold (M, D) via $\varphi_* \times \varphi_*$. This covariant derivative will be denoted simply by $\varphi^\# D$ (instead of $(\varphi_* \times \varphi_*)^\# D$). It is given by

$$(\varphi^\# D)_\xi \widetilde{Y} := \varphi_\#^{-1} D_{(\varphi_*)_\#\xi}\varphi_\#\widetilde{Y}; \qquad \xi \in \mathfrak{X}(TM), \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)\ ,$$

or, equivalently, $\quad D_{(\varphi_*)_\#\xi}\varphi_\#\widetilde{Y} = (\varphi_* \times \varphi_*) \circ (\varphi^\# D)_\xi \widetilde{Y} \circ \varphi_*^{-1}$.
If $\varphi^\# D = D$, and hence $\varphi_\#(D_\xi \widetilde{Y}) = D_{(\varphi_*)_\#\xi}\varphi_\#\widetilde{Y}$ for all $\widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$, then φ is called an *automorphism* of D. We denote by $\mathrm{Aut}(D)$ the group of all automorphisms of D. The following observation can be checked by an immediate calculation.

Lemma 5.2 *Let (M, D) be a line element D-manifold, φ a diffeomorphism of M and $\varphi^\# D$ the pull-back of D. Let $\mu^\#$ and $(\mu^\mathsf{v})^\#$ denote the deflection and v-deflection of $\varphi^\# D$. Then $\mu^\# = \varphi_\#^{-1} \circ \mu \circ (\varphi_*)_\#$ and $(\mu^\mathsf{v})^\# = \varphi_\#^{-1} \circ \mu^\mathsf{v} \circ \varphi_\#$. If, in particular, D is regular (or strongly regular), then $\varphi^\# D$ is also regular (or strongly regular). We have for all $\varphi \in \mathrm{Aut}(D)$*

$$\varphi_\# \circ \mu = \mu \circ (\varphi_*)_\# \quad \textit{and} \tag{11}$$
$$\varphi_\# \circ \mu^\mathsf{v} = \mu^\mathsf{v} \circ \varphi_\#\ . \tag{12}$$

Lemma 5.3 *Let (M,D) be a line element D-manifold, $\varphi \in \mathrm{Diff}(M)$. If $T(\varphi^{\#}D)$ denotes the torsion of $\varphi^{\#}D$, then*

$$\varphi_{\#}\left(T(\varphi^{\#}D)(\xi,\eta)\right) = T(D)\left((\varphi_*)_{\#}\xi, (\varphi_*)_{\#}\eta\right), \qquad \xi,\eta \in \mathfrak{X}(TM).$$

In particular, if $\varphi \in \mathrm{Aut}(D)$, then the torsion of D is invariant under φ:

$$\varphi_{\#} \circ T(D) = T(D) \circ \left((\varphi_*)_{\#} \times (\varphi_*)_{\#}\right) .$$

Proof.

$$\begin{aligned}
\varphi_{\#}\left(T(\varphi^{\#}D)(\xi,\eta)\right) &= \varphi_{\#}\left((\varphi^{\#}D)_{\xi}\mathbf{j}\eta - (\varphi^{\#}D)_{\eta}\mathbf{j}\xi - \mathbf{j}\,[\xi,\eta]\right) = \\
&= D_{(\varphi_*)_{\#}\xi}\varphi_{\#}(\mathbf{j}\eta) - D_{(\varphi_*)_{\#}\eta}\varphi_{\#}(\mathbf{j}\xi) - \mathbf{j}\,[(\varphi_*)_{\#}\xi, (\varphi_*)_{\#}\eta] = \\
&= D_{(\varphi_*)_{\#}\xi}\mathbf{j}((\varphi_*)_{\#}\eta) - D_{(\varphi_*)_{\#}\eta}\mathbf{j}((\varphi_*)_{\#}\xi) - \mathbf{j}\,[(\varphi_*)_{\#}\xi, (\varphi_*)_{\#}\eta] = \\
&= T(D)\left((\varphi_*)_{\#}\xi, (\varphi_*)_{\#}\eta\right).
\end{aligned}$$

□

Proposition 5.4 *Let (M,D) be a line element D-manifold and $\varphi \in \mathrm{Diff}(M)$. φ is an automorphism of D if and only if for every curve $c\colon I \to TM$ whose velocity field is extendible we have*

$$D_{\varphi_*\circ c}(\varphi_{\#}\widetilde{Y}) \circ (\varphi_* \circ c) = (\varphi_* \times \varphi_*) D_c(\widetilde{Y} \circ c) , \quad \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau). \tag{13}$$

Proof. *(a)* Let φ be an automorphism of D. Consider a curve $c\colon I \to TM$, and suppose that there exists a vector field ξ defined in a neighbourhood of $\mathrm{Im}(c) \subset TM$ such that $\dot{c} = \xi \circ c$. Then

$$\begin{aligned}
\dot{\overline{\varphi_* \circ c}} &= \varphi_{**} \circ \dot{c} = \varphi_{**} \circ \xi \circ c = \\
&= \varphi_{**} \circ \xi \circ \varphi_*^{-1} \circ \varphi_* \circ c = (\varphi_*)_{\#}\xi \circ \varphi_* \circ c,
\end{aligned}$$

so for all $t \in I$ we have

$$\begin{aligned}
(D_{\varphi_*\circ c}(\varphi_{\#}\widetilde{Y}) \circ (\varphi_* \circ c))(t) &= D_{\dot{\overline{\varphi_*\circ c}}(t)}\varphi_{\#}\widetilde{Y} = \\
&= D_{(\varphi_*)_{\#}\xi\circ(\varphi_*\circ c)(t)}\varphi_{\#}\widetilde{Y} = (D_{(\varphi_*)_{\#}\xi}(\varphi_{\#}\widetilde{Y}))(\varphi_* \circ c)(t) \overset{\text{cond.}}{=} \\
&= (\varphi_* \times \varphi_*) \circ D_{\xi}\widetilde{Y} \circ \varphi_*^{-1} \circ \varphi_* \circ c(t) = (\varphi_* \times \varphi_*) D_{\xi}\widetilde{Y}(c(t)) = \\
&= (\varphi_* \times \varphi_*) D_{\xi(c(t))}\widetilde{Y} = (\varphi_* \times \varphi_*)(D_{\dot{c}(t)}\widetilde{Y}) = \\
&= (\varphi_* \times \varphi_*)(D_c(\widetilde{Y} \circ c))(t);
\end{aligned}$$

thus relation (13) is valid.
(b) Conversely, suppose relation (13) is true. It is enough to show that for all $z \in TTM$, $\widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$ we have

$$(\varphi_* \times \varphi_*) D_z\widetilde{Y} = D_{\varphi_{**}(z)}\varphi_{\#}\widetilde{Y} .$$

If $z \in T_vTM$, choose a vector field $\xi \in \mathfrak{X}(TM)$ such that $\xi(v) = z$. Let $c\colon I \to TM$ be the integral curve of ξ starting from v. Then

$$\xi \circ c = \dot{c}, \quad z = \xi(v) = \xi(c(0)) = \dot{c}(0) ,$$

so

$$\begin{aligned}(\varphi_* \times \varphi_*)D_z\widetilde{Y} = (\varphi_* \times \varphi_*)D_{\dot{c}(0)}\widetilde{Y} = (\varphi_* \times \varphi_*)(D_c(\widetilde{Y} \circ c))(0) \overset{(13)}{=} \\ = (D_{\varphi_* \circ c}(\varphi_\#\widetilde{Y}) \circ (\varphi_* \circ c))(0) = D_{\dot{\overline{\varphi_* \circ c}}(0)}\varphi_\#\widetilde{Y} = \\ = D_{\varphi_{**}(\dot{c}(0))}\varphi_\#\widetilde{Y} = D_{\varphi_{**}(z)}\varphi_\#\widetilde{Y} ,\end{aligned}$$

as was to be shown. □

Corollary 5.5 *If (M, D) is a line element D-manifold, $\varphi \in \mathrm{Aut}(D)$ and $\gamma\colon I \to M$ is a curve whose acceleration field is extendible, then*

$$D_{\dot{\overline{\varphi \circ \gamma}}}(\varphi_\#\widetilde{Y}) \circ \dot{\overline{\varphi \circ \gamma}} = (\varphi_* \times \varphi_*)D_{\dot{\gamma}}(\widetilde{Y} \circ \dot{\gamma}). \tag{14}$$

Theorem 5.6 *Let (M, D) be a regular line element D-manifold, $\mathcal{H}_D$ be the Ehresmann connection induced by D, $\varphi \in \mathrm{Diff}(M)$. φ is an automorphism of D if and only if the following three conditions are satisfied:*
a) φ is an automorphism of the induced Ehresmann connection $\mathcal{H}_D$,
b) $\varphi_\# \circ \mathcal{S} = \mathcal{S} \circ (\varphi_\# \times \varphi_\#)$,
c) $\varphi_\# \circ \mathcal{P} = \mathcal{P} \circ (\varphi_\# \times \varphi_\#)$.

Proof. Suppose that $\varphi \in \mathrm{Aut}(D)$. Then for any vector field X on M we have

$$\begin{aligned}\varphi_{**} \circ \mathcal{H}_D \circ (\varphi_*^{-1} \times \varphi_*^{-1})(\widehat{X}) = \varphi_{**} \circ \mathcal{H}_D \circ (\varphi_*^{-1}, \varphi_*^{-1} \circ X \circ \tau) = \\ = \varphi_{**} \circ \mathcal{H}_D \circ (\varphi_*^{-1}, \varphi_*^{-1} \circ X \circ \varphi \circ \varphi^{-1} \circ \tau) = \\ = \varphi_{**} \circ \mathcal{H}_D \circ (\varphi_*^{-1}, \varphi_\#^{-1}X \circ \tau \circ \varphi_*^{-1}) = \\ = \varphi_{**} \circ \mathcal{H}_D \circ \widehat{\varphi_\#^{-1}X} \circ \varphi_*^{-1} = (\varphi_*)_\# \circ \mathcal{H}_D \circ \widehat{\varphi_\#^{-1}X} = \\ = (\varphi_*)_\# \circ \Big((\varphi_\#^{-1}X)X^{\mathsf{c}} - \mathbf{i}(\mu^{\mathsf{v}})^{-1}D_{(\varphi_\#^{-1}X)^{\mathsf{c}}}\delta\Big) = \\ = (\varphi_*)_\# \Big((\varphi_*^{-1})_\# X^{\mathsf{c}} - \mathbf{i}(\mu^{\mathsf{v}})^{-1}D_{(\varphi_*^{-1})_\# X^{\mathsf{c}}}\varphi_\#^{-1}\delta\Big) \overset{\text{cond.}}{=} \\ = X^{\mathsf{c}} - (\varphi_*)_\# \circ \mathbf{i} \circ (\mu^{\mathsf{v}})^{-1} \circ \varphi_\#^{-1}D_{X^{\mathsf{c}}}\delta = \\ = X^{\mathsf{c}} - \mathbf{i} \circ \varphi_\# \circ (\mu^{\mathsf{v}})^{-1} \circ \varphi_\#^{-1} \circ D_{X^{\mathsf{c}}}\delta \overset{(12)}{=} \\ = X^{\mathsf{c}} - \mathbf{i} \circ (\mu^{\mathsf{v}})^{-1} \circ \varphi_\# \circ \varphi_\#^{-1} \circ D_{X^{\mathsf{c}}}\delta = \mathcal{H}_D(\widehat{X}),\end{aligned}$$

hence $\varphi \in \mathrm{Aut}(\mathcal{H}_D)$, so a) is true.

Now we check that $\mathcal{S}$ and $\mathcal{P}$ are invariant under φ. Let X and Y be vector fields on M. Then, on the one hand,

$$\varphi_\#(\mathcal{S}(\widehat{X},\widehat{Y})) = -\varphi_\#(D_{Y^\mathsf{v}}\widehat{X}) - \varphi_\# \circ \mathbf{j}[X^\mathsf{h}, Y^\mathsf{v}] \overset{\text{cond.}}{=}$$
$$= -D_{(\varphi_*)_\# Y^\mathsf{v}}\varphi_\#\widehat{X} - \mathbf{j}[(\varphi_*)_\# X^\mathsf{h}, (\varphi_*)_\# X^\mathsf{v}] \overset{\text{a)}}{=}$$
$$= -D_{(\varphi_\# Y)^\mathsf{v}}\widehat{\varphi_\# X} - \mathbf{j}[(\varphi_\# X)^\mathsf{h}, (\varphi_\# X)^\mathsf{v}] =$$
$$= \mathcal{S}(\widehat{\varphi_\# X}, \widehat{\varphi_\# Y}) = \mathcal{S}(\varphi_\#\widehat{X}, \varphi_\#\widehat{Y}) \ .$$

On the other hand,

$$\varphi_\#(\mathcal{P}(\widehat{X},\widehat{Y})) = \varphi_\#(D_{X^\mathsf{v}}Y) - \varphi_\# \circ \mathcal{V}[X^\mathsf{h}, Y^\mathsf{v}] \overset{\text{cond.}}{=}$$
$$= D_{(\varphi_*)_\# X^\mathsf{h}}\varphi_\#\widehat{Y} - \varphi_\# \circ \mathcal{V}[X^\mathsf{h}, Y^\mathsf{v}] \overset{\text{a)}}{=}$$
$$= D_{(\varphi_\# X)^\mathsf{h}}\widehat{\varphi_\# Y} - \mathcal{V}[(\varphi_*)_\# X^\mathsf{h}, (\varphi_*)_\# Y^\mathsf{v}] =$$
$$= D_{(\varphi_\# X)^\mathsf{h}}\varphi_\#\widehat{Y} - \mathcal{V}[(\varphi_\# X)^\mathsf{h}, (\varphi_\# Y)^\mathsf{v}] =$$
$$= \mathcal{P}(\widehat{\varphi_\# X}, \widehat{\varphi_\# Y}) = \mathcal{P}(\varphi_\#\widehat{X}, \varphi_\#\widehat{Y}),$$

as we claimed.

Conversely, suppose that conditions a), b) and c) are satisfied. Let X be a vector field on M. Then

$$\varphi_\# \circ \mathcal{S}(\widehat{X},\widehat{Y}) = -\varphi_\#(D_{Y^\mathsf{v}}\widehat{X}) - \varphi_\# \circ \mathbf{j}[X^\mathsf{h}, Y^\mathsf{v}] =$$
$$= -\varphi_\#(D_{Y^\mathsf{v}}\widehat{X}) - \mathbf{j}[(\varphi_*)_\# X^\mathsf{h}, (\varphi_*)_\# Y^\mathsf{v}],$$

and

$$\mathcal{S} \circ (\varphi_\#\widehat{X}, \varphi_\#\widehat{Y}) = -D_{(\varphi_\# Y)^\mathsf{v}}\varphi_\#\widehat{X} - \mathbf{j}[(\varphi_\# X)^\mathsf{h}, (\varphi_\# Y)^\mathsf{v}] \overset{\text{a)}}{=}$$
$$= -D_{(\varphi_*)_\# Y^\mathsf{v}}\varphi_\#\widehat{X} - \mathbf{j}[(\varphi_*)_\# X^\mathsf{h}, (\varphi_*)_\# Y^\mathsf{v}],$$

so by condition b) we have

$$(*) \qquad \varphi_\#(D_{Y^\mathsf{v}}\widehat{X}) = D_{(\varphi_*)_\# Y^\mathsf{v}}\varphi_\#\widehat{X}.$$

Similarly,

$$\varphi_\# \circ \mathcal{P}(\widehat{X},\widehat{Y}) = \varphi_\#(D_{X^\mathsf{h}}\widehat{Y}) - \varphi_\# \circ \mathcal{V}[X^\mathsf{h}, Y^\mathsf{v}] \overset{\text{a)}}{=}$$
$$= \varphi_\#(D_{X^\mathsf{h}}\widehat{Y}) - \mathcal{V}[(\varphi_*)_\# X^\mathsf{h}, (\varphi_*)_\# Y^\mathsf{v}],$$

and

$$\mathcal{P}(\varphi_\#\widehat{X}, \varphi_\#\widehat{Y}) = D_{(\varphi_\# X)^\mathsf{h}}\varphi_\#\widehat{Y} - \mathcal{V}[(\varphi_\# X)^\mathsf{h}, (\varphi_\# Y)^\mathsf{v}] \overset{\text{cond.}}{=}$$
$$= D_{(\varphi_*)_\# X^\mathsf{h}}\varphi_\#\widehat{Y} - \mathcal{V}[(\varphi_*)_\# X^\mathsf{h}, (\varphi_*)_\# Y^\mathsf{v}],$$

hence condition c) implies

$$(**) \qquad \varphi_\#(D_{X^\mathsf{h}}\widehat{Y}) = D_{(\varphi_*)_\# X^\mathsf{h}}\varphi_\#\widehat{Y}.$$

From $(*)$ and $(**)$ it follows that

$$(+) \qquad \varphi_{\#}(D_\xi \widehat{Y}) = D_{(\varphi_*)_{\#}\xi}\varphi_{\#}\widehat{Y}$$

for all $\xi \in \mathfrak{X}(TM)$, $Y \in \mathfrak{X}(M)$, thus the invariance of D under φ is valid over the basic vector fields. To complete our argument, we have to show that for any function $F \in C^\infty(TM)$,

$$(++) \qquad \varphi_{\#}(D_\xi(F\widehat{Y})) = D_{(\varphi_*)_{\#}\xi}\varphi_{\#}F\widehat{Y} \ .$$

The left-hand side of $(++)$ can be transformed as follows:

$$\begin{aligned} \varphi_{\#}(D_\xi(F\widehat{Y})) &= \varphi_{\#}((\xi F)\widehat{Y} + FD_\xi\widehat{Y}) = \\ &= ((\xi F)\circ\varphi_*^{-1})\varphi_{\#}\widehat{Y} + (F\circ\varphi_*^{-1})\varphi_{\#}(D_\xi\widehat{Y}), \end{aligned}$$

while the right-hand side of $(++)$ is

$$D_{(\varphi_*)_{\#}\xi}\varphi_{\#}(F\widehat{Y}) = (\varphi_*)_{\#}\xi(F\circ\varphi_*^{-1})\varphi_{\#}\widehat{Y} + (F\circ\varphi_*^{-1})D_{(\varphi_*)_{\#}\xi}\varphi_{\#}\widehat{Y}.$$

Since

$$(\xi F)\circ\varphi_*^{-1} = (\varphi_*)_{\#}\xi(F\circ\varphi_*^{-1}),$$

we obtain the desired equality. □

Proposition 5.7 *If (M, D) is a regular line element D-manifold, then*

$$(15) \qquad \mathrm{Aut}(D) \subset \mathrm{Aff}(D).$$

Proof. Let $\gamma\colon I \to M$ be a geodesic of D. Then $D_{\ddot{\gamma}(t)}\delta = 0$ for all $t \in I$. Let φ be an automorphism of D. We show that

$$D_{\ddot{\overline{\varphi\circ\gamma}}(t)}\delta = 0 \ , \qquad t \in I \ ;$$

hence $\varphi\circ\gamma$ is also a geodesic of D, therefore $\varphi \in \mathrm{Aff}(D)$.

Let $\mathbf{h}$ be the horizontal projector associated to $\mathcal{H}_D$. Then $\mathbf{h}\circ\ddot{\gamma} = \ddot{\gamma}$ (since γ is also a geodesic of $\mathcal{H}_D$ by Lemma 5.1), and hence

$$\varphi_{**}\circ\ddot{\gamma}(t) = \varphi_{**}\circ\mathbf{h}\circ\ddot{\gamma}(t) \overset{5.6}{=} \mathbf{h}\circ\varphi_{**}\circ\ddot{\gamma}(t) \ .$$

Thus

$$\begin{aligned} D_{\ddot{\overline{\varphi\circ\gamma}}(t)}\delta = D_{\varphi_{**}\circ\ddot{\gamma}(t)}\delta = D_{\mathbf{h}\circ\varphi_{**}\circ\ddot{\gamma}(t)}\delta = D_{\mathcal{H}_D\circ\mathbf{j}\circ\ddot{\overline{\varphi\circ\gamma}}(t)}\delta = \\ = (D\delta\circ\mathcal{H}_D)(\mathbf{j}\circ\ddot{\overline{\varphi\circ\gamma}}(t)) = \mu^{\mathcal{H}_D}(\mathbf{j}\circ\ddot{\overline{\varphi\circ\gamma}}(t)) \overset{4.1}{=} 0 \ , \end{aligned}$$

since the $\mathcal{H}_D$-deflection of D vanishes by Theorem 4.1. □

Theorem 5.8 *Let (M, D) be a regular line element D-manifold and $\varphi \in \mathrm{Diff}(M)$. $\varphi \in \mathrm{Aut}(D)$ if and only if the following four conditions are satisfied:*

a) $\varphi \in \mathrm{Aff}(D)$,

b) $\varphi_{\#} \circ \mathbf{T^s} = \mathbf{T^s} \circ \varphi_{\#}$,

c) $\varphi_{\#} \circ \mathcal{S} = \mathcal{S} \circ (\varphi_{\#} \times \varphi_{\#})$,

d) $\varphi_{\#} \circ \mathcal{P} = \mathcal{P} \circ (\varphi_{\#} \times \varphi_{\#})$

($\mathbf{T^s}$ is the strong torsion of the induced Ehresmann connection $\mathcal{H}_D$).

Proof. (1) Suppose that $\varphi \in \mathrm{Aut}(D)$. Then, by 5.6, conditions c) and d) are satisfied and $\varphi \in \mathrm{Aut}(\mathcal{H}_D)$. By 3.3, we have condition b) and $\varphi \in \mathrm{Aut}(S_{\mathcal{H}_D})$. Finally,

$$\varphi \in \mathrm{Aut}(S_{\mathcal{H}_D}) \overset{3.1}{\iff} \varphi \in \mathrm{Aff}(S_{\mathcal{H}_D}) \overset{3.2}{\iff} \varphi \in \mathrm{Aff}(\mathcal{H}_D) \overset{(10)}{\iff} \varphi \in \mathrm{Aff}(D),$$

so we get condition a).
(2) Conversely, we suppose that relations a)–d) hold. Then $\varphi \in \mathrm{Aff}(D)$ (condition a)) is equivalent to $\varphi \in \mathrm{Aff}(S_{\mathcal{H}_D})$. By $\varphi \in \mathrm{Aff}(S_{\mathcal{H}_D})$, condition b) and Theorem 3.3 we have $\varphi \in \mathrm{Aut}(\mathcal{H}_D)$. By $\varphi \in \mathrm{Aut}(\mathcal{H}_D)$, conditions c) and d) and Theorem 5.6 it follows that $\varphi \in \mathrm{Aut}(D)$. □

Corollary 5.9 *Let (M, D) be a regular line element D-manifold and $\varphi \in \mathrm{Diff}(M)$. φ is an automorphism of D if and only if the following conditions are satisfied:*

a) $\varphi \in \mathrm{Aff}(D)$,

b') $\varphi_{\#} \circ i_\delta \mathcal{T} = i_\delta \mathcal{T} \circ \varphi_{\#}$,

c) $\varphi_{\#} \circ \mathcal{S} = \mathcal{S} \circ (\varphi_{\#} \times \varphi_{\#})$,

d) $\varphi_{\#} \circ \mathcal{P} = \mathcal{P} \circ (\varphi_{\#} \times \varphi_{\#})$.

Proof. We have only to prove that condition *b)* in 5.8 is equivalent to condition *b')*. First suppose that condition b) (and d)) of Theorem 5.8 hold. Then for every vector fields $X, Y \in \mathfrak{X}(M)$,

$$\varphi_{\#} \circ i_\delta \mathcal{T}(\widehat{X}) \overset{(7)}{=} \varphi_{\#} \circ (\mathbf{T^s}(\widehat{X}) + i_\delta \mathcal{P}(\widehat{X})) \overset{\text{cond.}}{=}$$
$$= \mathbf{T^s}(\varphi_{\#}\widehat{X}) + \mathcal{P}(\varphi_{\#}\delta, \varphi_{\#}\widehat{X}) \overset{(2)}{=} \mathbf{T^s}(\varphi_{\#}\widehat{X}) + i_\delta \mathcal{P}(\varphi_{\#}\widehat{X}) \overset{(7)}{=} i_\delta \mathcal{T} \circ \varphi_{\#}(\widehat{X});$$

so we have condition *b')*.
Conversely, if conditions b') (and d)) of Theorem 5.9 are valid, then for any vector field $X \in \mathfrak{X}(M)$,

$$\varphi_{\#}\mathbf{T}^{\mathsf{s}}(\widehat{X}) \overset{(7)}{=} \varphi_{\#}(i_\delta \mathcal{T}(\widehat{X}) - i_\delta \mathcal{P}(\widehat{X})) \overset{b')}{=}$$
$$= i_\delta \mathcal{T}(\varphi_{\#}\widehat{X}) - i_\delta \mathcal{P}(\varphi_{\#}\widehat{X}) = \mathbf{T}^{\mathsf{s}} \circ \varphi_{\#}(\widehat{X}),$$

so we get 5.8(b). □

Theorem 5.10 *Let (M, D) be a regular line element D-manifold. A diffeomorphism $\varphi\colon M \to M$ is an automorphism of the covariant derivative D if and only if the following conditions are satisfied:*

A) $\varphi \in \mathrm{Aff}(D)$,

B) $\varphi_{\#} \circ T^{\mathsf{v}}(D) = T^{\mathsf{v}}(D) \circ ((\varphi_*)_{\#} \times (\varphi_*)_{\#})$,

C) $\varphi_{\#} \circ T(D) = T(D) \circ ((\varphi_*)_{\#} \times (\varphi_*)_{\#})$.

Proof. Necessity. Suppose that $\varphi \in \mathrm{Aut}(D)$. Then $\varphi \in \mathrm{Aff}(D)$. We show that conditions B) and C) hold. To do this, we evaluate the left-hand side and the right-hand side of these relations on pairs of the form $(X^{\mathsf{v}}, Y^{\mathsf{v}})$, $(X^{\mathsf{v}}, Y^{\mathsf{h}})$ and $(X^{\mathsf{h}}, Y^{\mathsf{h}})$ where $X, Y \in \mathfrak{X}(M)$.
Checking of B)

$$\begin{aligned}
\varphi_{\#} \circ T(D)(X^{\mathsf{v}}, Y^{\mathsf{v}}) =& \varphi_{\#}(D_{X^{\mathsf{v}}}\mathbf{j}Y^{\mathsf{v}} - D_{Y^{\mathsf{v}}}\mathbf{j}X^{\mathsf{v}} - \mathbf{j}[X^{\mathsf{v}}, Y^{\mathsf{v}}]) = 0,\\
T(D)((\varphi_*)_{\#}X^{\mathsf{v}}, (\varphi_*)_{\#}Y^{\mathsf{v}}) =& D_{(\varphi_*)_{\#}X^{\mathsf{v}}}\mathbf{j}(\varphi_*)_{\#}Y^{\mathsf{v}} - D_{(\varphi_*)_{\#}Y^{\mathsf{v}}}\mathbf{j}(\varphi_*)_{\#}X^{\mathsf{v}} -\\
&- \mathbf{j}[(\varphi_*)_{\#}X^{\mathsf{v}}, (\varphi_*)_{\#}Y^{\mathsf{v}}] = D_{(\varphi_{\#}X)^{\mathsf{v}}}\mathbf{j}(\varphi_{\#}Y)^{\mathsf{v}} -\\
&- D_{(\varphi_{\#}Y)^{\mathsf{v}}}\mathbf{j}(\varphi_{\#}X)^{\mathsf{v}} - \mathbf{j}[(\varphi_{\#}X)^{\mathsf{v}}, (\varphi_{\#}Y)^{\mathsf{v}}] = 0;
\end{aligned}$$

$$\begin{aligned}
\varphi_{\#} \circ T(D)(X^{\mathsf{v}}, Y^{\mathsf{h}}) =& \varphi_{\#}(D_{X^{\mathsf{v}}}\widehat{Y} - \mathbf{j}[X^{\mathsf{v}}, Y^{\mathsf{h}}]) = \varphi_{\#}D_{X^{\mathsf{v}}}\widehat{Y} \overset{\text{cond.}}{=}\\
&= D_{(\varphi_*)_{\#}X^{\mathsf{v}}}\varphi_{\#}\widehat{Y},\\
T(D)((\varphi_*)_{\#}X^{\mathsf{v}}, (\varphi_*)_{\#}Y^{\mathsf{h}}) =& D_{(\varphi_*)_{\#}X^{\mathsf{v}}}\mathbf{j} \circ (\varphi_*)_{\#}Y^{\mathsf{h}} - D_{(\varphi_*)_{\#}Y^{\mathsf{h}}}\mathbf{j}(\varphi_*)_{\#}X^{\mathsf{v}} -\\
&- \mathbf{j}[(\varphi_*)_{\#}X^{\mathsf{v}}, (\varphi_*)_{\#}Y^{\mathsf{h}}] \overset{5.6}{=} D_{(\varphi_*)_{\#}X^{\mathsf{v}}}\varphi_{\#}\widehat{Y} -\\
&- \mathbf{j}[(\varphi_{\#}X)^{\mathsf{v}}, (\varphi_{\#}Y)^{\mathsf{h}}] = D_{(\varphi_*)_{\#}X^{\mathsf{v}}}\varphi_{\#}\widehat{Y}.
\end{aligned}$$

$$\begin{aligned}
\varphi_{\#} \circ T(D)(X^{\mathsf{h}}, Y^{\mathsf{h}}) =& \varphi_{\#}(D_{X^{\mathsf{h}}}\widehat{Y} - D_{Y^{\mathsf{h}}}\widehat{X} - \mathbf{j}[X^{\mathsf{h}}, Y^{\mathsf{h}}]) \overset{\text{cond.}}{=}\\
&= D_{(\varphi_*)_{\#}X^{\mathsf{h}}}\varphi_{\#}\widehat{Y} - D_{(\varphi_*)_{\#}Y^{\mathsf{h}}}\varphi_{\#}\widehat{X} -\\
&- \mathbf{j}[(\varphi_*)_{\#}X^{\mathsf{h}}, (\varphi_*)_{\#}Y^{\mathsf{h}}]\ ,\\
T(D)((\varphi_*)_{\#}X^{\mathsf{h}}, (\varphi_*)_{\#}Y^{\mathsf{h}}) =& D_{(\varphi_*)_{\#}X^{\mathsf{h}}}\mathbf{j} \circ (\varphi_*)_{\#}Y^{\mathsf{h}} - D_{(\varphi_*)_{\#}Y^{\mathsf{h}}}\mathbf{j} \circ (\varphi_*)_{\#}X^{\mathsf{h}} -\\
&- \mathbf{j}[(\varphi_*)_{\#}X^{\mathsf{h}}, (\varphi_*)_{\#}Y^{\mathsf{h}}] = D_{(\varphi_*)_{\#}X^{\mathsf{h}}}\widehat{Y} -\\
&- D_{(\varphi_*)_{\#}Y^{\mathsf{h}}}\widehat{X} - \mathbf{j}[(\varphi_*)_{\#}X^{\mathsf{h}}, (\varphi_*)_{\#}Y^{\mathsf{h}}].
\end{aligned}$$

Checking of C)

$$\begin{aligned}
\varphi_\# T^{\mathsf{v}}(D)(X^{\mathsf{v}},Y^{\mathsf{v}}) =& \varphi_\#(D_{X^{\mathsf{v}}}\mathcal{V}Y^{\mathsf{v}} - D_{Y^{\mathsf{v}}}\mathcal{V}X^{\mathsf{v}} - \mathcal{V}[X^{\mathsf{v}},Y^{\mathsf{v}}]) \overset{\text{cond.}}{=} \\
&= D_{(\varphi_*)_\# X^{\mathsf{v}}}\varphi_\#\widehat{Y} - D_{(\varphi_*)_\# Y^{\mathsf{v}}}\widehat{X}, \\
T^{\mathsf{v}}(D)((\varphi_*)_\# X^{\mathsf{v}},(\varphi_*)_\# Y^{\mathsf{v}}) =& D_{(\varphi_*)_\# X^{\mathsf{v}}}\mathcal{V}(\varphi_*)_\# Y^{\mathsf{v}} - D_{(\varphi_*)_\# Y^{\mathsf{v}}}\mathcal{V}(\varphi_*)_\# X^{\mathsf{v}} - \\
&- \mathcal{V}[(\varphi_*)_\# X^{\mathsf{v}},(\varphi_*)_\# Y^{\mathsf{v}}] = D_{(\varphi_*)_\# X^{\mathsf{v}}}\mathcal{V}\circ\mathbf{i}\circ\varphi_\# Y - \\
&- D_{(\varphi_*)_\# Y^{\mathsf{v}}}\mathcal{V}\circ\mathbf{i}\circ\varphi_\# X - \mathcal{V}[(\varphi_\# X)^{\mathsf{v}},(\varphi_\# Y)^{\mathsf{v}}] = \\
&= D_{(\varphi_*)_\# X^{\mathsf{v}}}\varphi_\#\widehat{Y} - D_{(\varphi_*)_\# Y^{\mathsf{v}}}\widehat{X}.
\end{aligned}$$

$$\begin{aligned}
\varphi_\# \circ T^{\mathsf{v}}(D)(X^{\mathsf{v}},Y^{\mathsf{h}}) =& \varphi_\#(-D_{Y^{\mathsf{h}}}\widehat{X} - \mathcal{V}[X^{\mathsf{v}},Y^{\mathsf{h}}]) \overset{5.6,\ \text{cond.}}{=} \\
&= -D_{(\varphi_*)_\# Y^{\mathsf{h}}}\varphi_\#\widehat{X} - \mathcal{V}[(\varphi_*)_\# X^{\mathsf{v}},(\varphi_*)_\# Y^{\mathsf{h}}], \\
T^{\mathsf{v}}(D)((\varphi_*)_\# X^{\mathsf{v}},(\varphi_*)_\# Y^{\mathsf{h}}) =& D_{(\varphi_*)_\# X^{\mathsf{v}}}\mathcal{V}\circ(\varphi_*)_\# Y^{\mathsf{h}} - D_{(\varphi_*)_\# Y^{\mathsf{h}}}\mathcal{V}\circ(\varphi_*)_\# X^{\mathsf{v}} - \\
&- \mathcal{V}[(\varphi_*)_\# X^{\mathsf{v}},(\varphi_*)_\# Y^{\mathsf{h}}] \overset{5.6}{=} -D_{(\varphi_*)_\# Y^{\mathsf{h}}}\varphi_\#\widehat{X} - \\
&- \mathcal{V}[(\varphi_*)_\# X^{\mathsf{v}},(\varphi_*)_\# Y^{\mathsf{h}}].
\end{aligned}$$

$$\begin{aligned}
\varphi_\# \circ T^{\mathsf{v}}(D)(X^{\mathsf{h}},Y^{\mathsf{h}}) =& \varphi_\#(-\mathcal{V}[X^{\mathsf{h}},Y^{\mathsf{h}}]) \overset{5.6}{=} -\mathcal{V}[(\varphi_*)_\# X^{\mathsf{h}},(\varphi_*)_\# Y^{\mathsf{h}}], \\
T^{\mathsf{v}}(D)((\varphi_*)_\# X^{\mathsf{h}},(\varphi_*)_\# Y^{\mathsf{h}}) =& D_{(\varphi_*)_\# X^{\mathsf{h}}}\mathcal{V}\circ(\varphi_*)_\# Y^{\mathsf{h}} - \\
&- D_{(\varphi_*)_\# Y^{\mathsf{h}}}\mathcal{V}\circ(\varphi_*)_\# X^{\mathsf{h}} - \mathcal{V}[(\varphi_*)_\# X^{\mathsf{h}},(\varphi_*)_\# Y^{\mathsf{h}}] = \\
&= -\mathcal{V}[(\varphi_*)_\# X^{\mathsf{h}},(\varphi_*)_\# Y^{\mathsf{h}}].
\end{aligned}$$

Sufficiency. Conditions A)-C) imply immediately that

$$\begin{aligned}
\varphi_\# \circ i_\delta \mathcal{T} &= i_\delta \mathcal{T}\circ\varphi_\#, \\
\varphi_\# \circ \mathcal{S} &= \mathcal{S}\circ(\varphi_\# \times \varphi_\#), \\
\varphi_\# \circ \mathcal{P} &= \mathcal{P}\circ(\varphi_\# \times \varphi_\#),
\end{aligned}$$

since all of the torsions $\mathcal{T}$, $\mathcal{S}$, $\mathcal{P}$ can be obtained from $T^{\mathsf{v}}(D)$ or $T(D)$ (see (3), (6), (8)), so, by 5.9, the sufficiency follows. □

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